Home Work Assignment #4, Problem 1,
Consider the general linear model \( Y = \eta + \varepsilon, \eta = X\beta, E[\varepsilon] = 0, Var[\varepsilon] = \sigma^2 V \), where \( V \) is a known positive definite matrix. Let \( M \) be an arbitrary, symmetric, positive definite matrix, let \( \tilde{\beta}_M \) denote an arbitrary solution to the following optimization problem:
\[
\min_{\beta} (Y - X\beta)'M(Y - X\beta) \tag{1}
\]
When \( X \) is not of full column rank, the optimal solution \( \tilde{\beta}_M \) is not unique.

The generalized least square estimator \( \hat{\beta}_{GLS} \) corresponds to \( M = V^{-1} \) and OLS estimator \( \hat{\beta} \) corresponds to \( M = I \).

Part (iii) Find conditions on the matrices \( X \) and \( V \), so that the OLS and the GLS estimators of \( \eta \) are equal [equivalently, the OLS and GLS of every estimable linear function of \( \beta \) are equal, since they all are a linear function of \( \eta \)].

The Result 4.5.2 addresses this problem for the full rank case. But the proof is incomplete.

**Result 4.5.2.** (pp.126, Dey and Ravishanker)
In the full rank model, \( \hat{\beta} \) and \( \hat{\beta}_{GLS} \) are identical if and only if \( \mathcal{C}(V^{-1}X) = \mathcal{C}(X) \).

Proof: In the full rank case \( \text{rank}(X) = p \), \( \hat{\beta} \) and \( \hat{\beta}_{GLS} \) unique are solutions to respective optimization criteria. The OLS and GLS are defined as solutions of normal equations
\[
(X'X)\beta = X'Y \quad \text{and} \quad (X'V^{-1}X)\beta = X'V^{-1}Y
\]
The two solutions are identical if and only if
\[
(X'X)^{-1}X'Y = (X'V^{-1}X)^{-1}X'V^{-1}Y \iff (X'V^{-1}X)(X'X)^{-1}X'Y = X'V^{-1}Y \tag{2}
\]
Now, let
\[
P_X = X(X'X)^{-1}X'Y \quad \text{denote the orthogonal projection matrix onto } \mathcal{C}(X).
\]
Using the unique orthogonal decomposition,
\[
Y = P_X Y + (I - P_X) Y = Y_1 + Y_2 \text{ (say)},
\]
note that, equation (2) is equivalent to [The remainder of the proof is missing in the book.]

\[
(X'X)^{-1}X'(Y_1 + Y_2) = (X'V^{-1}X)^{-1}X'V^{-1}(Y_1 + Y_2) \tag{3}
\]
Since, \( Y_i \in \mathcal{C}(X), Y_i = Xz \iff (X'X)^{-1}X'Y_i = (X'X)^{-1}X'Xz = Xz \tag{4}
\]
It is easy to see that for \( Y_i \in \mathcal{C}(X) \), value of the criterion (1) is 0 for all \( M \).

Hence \( \beta = z \) is optimal for all positive definite \( M \).

Therefore, \( 3 \) holds if and only if \( X'Y_2 = 0 \iff X'V^{-1}Y_2 = 0 \).

Hence, a necessary and sufficient condition for \( 3 \) to hold is that
\( X'x = 0 \Rightarrow X'V^{-1}x = 0 \), i.e., \( \mathcal{C}(V^{-1}X) \subset \mathcal{C}(X) \).

In addition, since \( V \) is non-singular, both these spaces have the same dimension. Therefore, \( \mathcal{C}(V^{-1}X) = \mathcal{C}(X) \).

**Note:** In the non-full rank model, we can’t assume the existence of \((X'X)^{-1}\) or \((X'V^{-1}X)^{-1}\), so OLS and GLS of \( \beta \) are not uniquely defined, hence equation (2) is not justifiable.

However, since the OLS and the GLS estimators of \( \eta \) are invariant to the choice of g-inverse, and all estimable functions are linear functions of \( \eta \), the extension of **Result 4.5.2 to non-full rank case must be stated in terms of the equality of** \( \hat{\eta} \) and \( \tilde{\eta}_{GLS} \), instead of \( \hat{\beta} \) and \( \tilde{\beta}_{GLS} \).

Instead of starting with (2) we start with

\[
X(X'X)^{-1}X'Y = X(X'V^{-1}X)^{-1}X'V^{-1}Y
\]

(2a)

The proof of the modified result for the non-full rank case follows along the lines above proof. Now, let

\[
P_x = X(X'X)^{-1}X'
\]

denote the orthogonal projection matrix onto \( \mathcal{C}(X) \).

Using the unique orthogonal decomposition,

\[
Y = P_xY + (I - P_x)Y \equiv Y_1 + Y_2 \text{ (say)},
\]

note that, equation (2a) is equivalent to

\[
X(X'X)^{-1}X'(Y_1 + Y_2) = X(X'V^{-1}X)^{-1}X'V^{-1}(Y_1 + Y_2)
\]

(3a)

It is easy to see that for \( Y_1 \in \mathcal{C}(X) \), value of the criterion (1) is 0 for all \( M \).

Hence \( \hat{Y}_i = Y_i \) is optimal for all positive definite \( M \).

Therefore, (3a) holds if and only if

\[
X(X'X)^{-1}X'Y_2 = X(X'V^{-1}X)^{-1}X'V^{-1}Y_2
\]

(4a)

However, the left side of (4a) equals \( P_xY_2 \), and \( P_x(I - P_x) = 0 \), it follows that \( P_xY_2 = 0 \).

Hence, a necessary and sufficient condition for (4a) to hold is that

\[X'x = 0 \Rightarrow X'V^{-1}x = 0, \text{ i.e., } \mathcal{C}(V^{-1}X) \subset \mathcal{C}(X).
\]

In addition, since \( V \) is non-singular, both these spaces have the same dimension. Therefore, the necessary and sufficient condition for \( \hat{\eta} = \tilde{\eta}_{GLS} \) is \( \mathcal{C}(V^{-1}X) = \mathcal{C}(X) \).

**Corollary:** \( \hat{\eta} \) and \( \tilde{\eta}_{GLS} \) are identical if and only if \( \mathcal{C}(VX) = \mathcal{C}(X) \).

**Proof:**

\[
\mathcal{C}(V^{-1}X) = \mathcal{C}(X) \Rightarrow \text{ For every } u \in \mathcal{C}(V^{-1}X),
\]

\[\exists \ a \ h \text{ such that } u = Xh. \Rightarrow Vu = VXh, \text{ and}
\]

\[Vu \in \mathcal{C}(X) \Rightarrow \mathcal{C}(X) \subset \mathcal{C}(VX).
\]

Similarly, we can show that \( \mathcal{C}(VX) \subset \mathcal{C}(X) \).

Therefore, \( \mathcal{C}(VX) = \mathcal{C}(X) \).
Remark: If the $X$ matrix contains a column of 1’s and the error variance-covariance matrix has intra-class correlation structure, $V = (1 - \rho)I + \rho J$, then $\hat{\eta}_{OLS} = \hat{\eta}_{GLS}$.

Note that

$$V1 = (1 - \rho)\mathbf{1} + \rho \mathbf{1}\mathbf{1}' = [1 + (n - 1)\rho] \mathbf{1} \in \mathcal{C}(X).$$

Furthermore, for an arbitrary column $x_i$ of $X$,

$$Vx_i = (1 - \rho)x_i + \rho \mathbf{1}'x_i = (1 - \rho)x_i + \rho (\sum_j x_j) \mathbf{1} \in \mathcal{C}(X).$$

Therefore, $\mathcal{C}(VX) = \mathcal{C}(X)$.

It follows from the above corollary that $\hat{\eta}_{OAS} = \hat{\eta}_{GLS}$. 