7.8. By Corollary 7.1.1 and Result 5.2.7, the least squares estimates of these parametric functions of $\beta$ have normal distributions. Also, $\frac{SSE}{(N-p)} \sim X^2_{N-p}$. The resulting $t$-intervals have the form in (7.4.3). The 95% C.I. for $\beta_1$ is $\hat{\beta}_1 \pm 4.73 \sqrt{\frac{V}{n_1}}$. For $\beta_2$, it is $\hat{\beta}_2 \pm 4.73 \sqrt{\frac{V}{n_2}}$. For $\beta_3$, it is $\hat{\beta}_3 \pm 4.73$. For $\beta_1 - \beta_2$, it is $\hat{\beta}_1 - \hat{\beta}_2 \pm 4.73 \sqrt{\frac{V}{n_1+n_2}}$. For $\beta_1 + \beta_3$, it is $\hat{\beta}_1 + \hat{\beta}_3 \pm 4.73 \sqrt{\frac{V}{n_1+n_2}}$.

7.11. Verify that
\[
\hat{\beta}_H = \frac{\sum_i \sum_j (X_{i,j} - \bar{X}_j)(Y_{i,j} - \bar{Y}_j)}{\sum_i \sum_j (X_{i,j} - \bar{X}_j)^2}, \quad \text{and} \quad \hat{\beta}_{0,H} = \bar{Y}_j - \hat{\beta}_H \bar{X}_j, \quad i = 1, 2.
\]

A point estimate of $\delta$ is then $\hat{\delta} = (\hat{\beta}_{2,0,H} - \hat{\beta}_{1,0,H})/\hat{\beta}_H$. Let $U = (\hat{\beta}_{2,0,H} - \hat{\beta}_{1,0,H}) - \hat{\beta}_H (\bar{X}_1 - \bar{X}_2 - \delta)$. Then, $U \sim N(0, \sigma^2)$, with
\[
\sigma^2 = \frac{\sum_i \sum_j (X_{i,j} - \bar{X}_j)^2}{\sum_i \sum_j (X_{i,j} - \bar{X}_j)^2}.
\]

Then, $SSE_H = \sum_i \sum_j (Y_{i,j} - \bar{Y}_j)^2 - \hat{\beta}_H (\bar{X}_1 - \bar{X}_2)^2$, and
\[
T = \frac{U/\sigma}{\sqrt{\frac{SSE_H/\sigma^2}{n_1+n_2}}}, \quad \text{under } H : \delta = 0. \quad \text{We can obtain the limits of the C.I. by solving the quadratic equation obtained by setting } T^2 = t_{n_1+n_2-3, \alpha/2}.
\]

7.18. $Y_0 = x_0^T \beta + \epsilon_0$, where $\epsilon_0 \sim N(0, \sigma^2)$. By the Gauss-Markov Theorem, $\hat{Y}_0 = x_0^T \hat{\beta} \sim N(x_0^T \beta, \sigma^2 x_0^T (X'X)^{-1} x_0)$.

(a) $E(Y_0) = E(x_0^T \beta + \epsilon_0) = x_0^T \beta$, and since $\epsilon_0$ is independent of $\epsilon$, $Var(Y_0 - \hat{Y}_0) = Var(Y_0) + Var(\hat{Y}_0) = \sigma^2 + \sigma^2 x_0^T (X'X)^{-1} x_0$. Replacing $\sigma^2$ by its least squares estimate gives the required result. We know that $(N-p)\sigma^2/\sigma^2 \sim X^2_{N-p}$. The distribution of the given statistic is therefore $t_{N-p}$.

(b) The 95% symmetric two-sided prediction interval for $Y_0$ is
\[
\{x_0^T \hat{\beta} \pm t_{N-p, 0.025} \sqrt{\sigma^2 (1 + x_0^T (X'X)^{-1} x_0)}\}.
\]

(c) The 100(1 - $\alpha$)% lower bound for $\gamma(\eta)$ is
\[
\gamma(\hat{\eta}) = z_0 \sqrt{\frac{\sigma^2}{1 + x_0^T (X'X)^{-1} x_0}},
\]
where $\hat{\gamma}(\eta) = x_0^T \beta + z_0 \sigma$, assuming $\sigma$ is known.

7.19. $H_0 : \beta_1 = 0$ implies $H_0 : \beta_1 = 0$. Clearly, $\hat{\beta}_{0,H} = \hat{\beta}_0$, $\hat{\beta}_{2,H} = \hat{\beta}_2$, while $\hat{\beta}_{1,H} = 0$. The test statistic $F_0 = (N-3)Q/SSE \sim F_{N-3, \alpha}$ under $H_0$, where $Q = \sum \hat{\beta}_i X_i (2 \hat{\beta}_0 + \hat{\beta}_1 X_1 + 2 \hat{\beta}_2 X_2^2)$, and $SSE = \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2^2)$. This is an approximation. Another way to find an approximate C.I. is to use the $\delta$-method.
8.2. Let \( y = (Y_1, \ldots, Y_N)' \), \( X = (1_N, X_1) \), where \( X_1 \) is an \( N \times k \) matrix, 
\( \beta = (\beta_0, \beta_1, \ldots, \beta_k)' \), and \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)' \) with \( \varepsilon \sim N_N(0, \Sigma^2 I) \). Now, \( H : \beta_i = d \) is equivalent to \( H : \beta_i = d \) where \( C' \beta = d \) where \( C' \) is a \((k+1)\)-dimensional vector with 1 in the \( j \)th position and zero elsewhere. Following the method in section 7.4.1, we obtain 
\[
\hat{\beta} = \left( Y - X' B^{-1} (X_1 y - NYX) \right) \frac{B^{-1} (X_1 y - NYX)}{B^{-1} (X_1 y - NYX)} ,
\]
where \( B = X_1 X_1 - NXX' \), so that \( C' \hat{\beta} - d = C' \beta - d \), and \( Q = (C' \beta - d)' / (C' \beta - d) \), \( a^{ij} \) being the \( i \)th diagonal element of \( (X'X)^{-1} \). Also, \( s = 1 \), and \( SSE = y'(I - X(X'X)^{-1}X'y) \), which has a chi-square distribution with \( N - k - 1 \) degrees of freedom. Then, by Result 7.4.1,
\[
F(H) = \frac{Q}{SSE/(N-k-1)} \sim F_{1, N-k-1}
\]
under \( H \).

8.5. The unrestricted least squares estimates of the parameters are obtained by minimizing \( \sum_{i=1}^{2} \sum_{i=1}^{n_i} (Y_{i,t} - \beta_{i,0} - \beta_{i,1} X_{i,t})^2 \) as \( \beta_{i,0} = Y_{i,t} - \hat{\beta}_{i,1} X_{i,t} \), and \( \beta_{i,1} = \left[ \sum_{i=1}^{n_i} (Y_{i,t} - \bar{Y}_{i,t}) (X_{i,t} - \bar{X}_{i,t}) \right] / \sum_{i=1}^{n_i} (X_{i,t} - \bar{X}_{i,t})^2, \) \( l = 1, 2 \). Also, the unrestricted error sum of squares is \( SSE = \sum_{i=1}^{2} \sum_{i=1}^{n_i} (Y_{i,t} - \bar{Y}_{i,t})^2 = \sum_{i=1}^{n_i} (X_{i,t} - \bar{X}_{i,t})^2 \).

(a) For a test of parallelism, the restricted least squares estimates and \( SSE_H \) under \( H : \beta_{i,1} = \beta_{2,1} = \beta_1 \), say, are obtained by minimizing \( \sum_{i=1}^{2} \sum_{i=1}^{n_i} (Y_{i,t} - \beta_{i,0} - \beta_1 X_{i,t})^2 \) as
\[
\hat{\beta}_{i,0,H} = Y_{i,t} - \hat{\beta}_{i,1,H} X_{i,t}, \quad l = 1, 2,
\]
\[
\hat{\beta}_{1,H} = \sum_{i=1}^{n_1} \sum_{i=1}^{n_i} (Y_{i,t} - \bar{Y}_{i,t}) (X_{i,t} - \bar{X}_{i,t}) / \sum_{i=1}^{n_1} (X_{i,t} - \bar{X}_{i,t})^2.
\]
Also,
\[
SSE_H = \sum_{i=1}^{n_1} \sum_{i=1}^{n_i} (Y_{i,t} - \bar{Y}_{i,t})^2 - \sum_{i=1}^{n_1} \sum_{i=1}^{n_i} (X_{i,t} - \bar{X}_{i,t})^2, \quad \text{and}
\]
\[
SSE_H - SSE = \sum_{i=1}^{n_1} \sum_{i=1}^{n_i} (X_{i,t} - \bar{X}_{i,t})^2 - \sum_{i=1}^{n_1} \sum_{i=1}^{n_i} (X_{i,t} - \bar{X}_{i,t})^2.
\]
The statistic
\[
F = (SSE_H - SSE) / (SSE/(N - 4)) \sim F_{1, N - 4}
\]
distribution under \( H \).
(b) We test the hypothesis that the two lines meet at a point on the y-axis, i.e., \( H : \beta_{1,0} = \beta_{2,0} = \beta_0, \) say. The restricted least squares estimates and \( SSE_H \) under \( H \) are obtained by minimizing
\[
\sum_{i=1}^{n_{11}} (Y_{i} - \beta_0 - \beta_{1,1} X_{i,1})^2
\]
and
\[
\sum_{i=1}^{n_{11}} (Y_{i} - \beta_0 - \beta_{1,1} X_{i,1})^2
\]
Also, the restricted error sum of squares is
\[
SSE_H = \sum_{i=1}^{n_{11}} (Y_{i} - \beta_0 - \beta_{1,1} X_{i,1})^2,
\]
and the test statistic \( F = (SSE_H - SSE)/(SSE/(N - 4)) \) has an \( F_{1,N-4} \) distribution under \( H \).

(c) The restricted least squares estimates and \( SSE_H \) under \( H : \beta_{1,0} = \beta_{2,0} = \beta_0; \beta_{1,1} = \beta_{2,1} = \beta_1, \) say, are obtained by minimizing
\[
\sum_{i=1}^{n_{11}} (Y_{i} - \beta_0 - \beta_{1,1} X_{i,1})^2
\]
as
\[
\beta_{0,H} = \frac{\sum_{i=1}^{n_{11}} (Y_{i} - \beta_0 - \beta_{1,1} X_{i,1})}{\sum_{i=1}^{n_{11}} X_{i,1}^2}, \quad \beta_{1,H} = \frac{\sum_{i=1}^{n_{11}} (Y_{i} - \beta_0 - \beta_{1,1} X_{i,1}) (X_{i,1} - \bar{X}_1)}{\sum_{i=1}^{n_{11}} (X_{i,1} - \bar{X}_1)^2}.
\]

The restricted error sum of squares is
\[
SSE_H = \sum_{i=1}^{n_{11}} (Y_{i} - \bar{Y}_1)^2 - \beta_{1,H} \sum_{i=1}^{n_{11}} (X_{i,1} - \bar{X}_1)^2.
\]
Then, \( F(H) = (N - 4)(SSE_H - SSE)/(2SSE) \sim F_{2,N-4} \) under \( H \).

8.6. (a) Let \( Y_k = \sum_{i=1}^{n_k} Y_{k,i}/n_k, X_k = \sum_{i=1}^{n_k} X_{k,i}/n_k, \sigma_{XX} = \sum_{i=1}^{n_k} (X_{k,i} - \bar{X}_k)^2, \) and \( \sigma_{XY} = \sum_{i=1}^{n_k} (Y_{k,i} - \bar{Y}_k)(X_{k,i} - \bar{X}_k) \) for \( k = 1, 2, \) the least squares estimates are \( \hat{\alpha}_k = \bar{Y}_k, \) \( \hat{\beta}_k = \bar{X}_k, \) and \( \hat{\beta} = |S_{XX}|^{-1} |S_{XY}|. \) The vertical distance between the lines is \( D = (\alpha_1 - \alpha_2) + \hat{\beta}(\bar{X}_1 - \bar{X}_2), \) with \( D = \bar{Y}_1 - \bar{Y}_2 - \hat{\beta}(\bar{X}_1 - \bar{X}_2). \) Since \( E(Y_k) = \alpha_k, \) for \( k = 1, 2, \) and \( E(\hat{\beta}) = \beta, \) we see that \( E(D) = D. \)

(b) We can verify that \( \hat{\beta} \sim N(\alpha_k, \sigma^2/n_k), k = 1, 2, \) and
\[
\hat{\beta} \sim N(\beta, \sigma^2/[S_{XX}^2 + S_{XY}^2]), \quad \sigma = \sqrt{(1/n_1 + 1/n_2 + (\bar{X}_1 - \bar{X}_2)^2)[S_{XX}^2 + S_{XY}^2])}. \]
An unbiased estimate of \( \sigma^2 \) is
\[
SSE/(N - 3), \] which is independent of \( D \) and \( D - D. \) A 95% symmetric C.I. for \( D \) is
\[
\hat{D} \pm [F_{1,n_1 + n_2 - 3, 0.05}]^{-1/2} \text{s.e.}(\hat{D}).
\]

9.10. If we impose the restrictions \( R^T \theta = 0, i = 1, \ldots, a \) and \( \mu^T = 0, \) we can solve the normal equations in the two-way nested model for \( \beta_{j(i)} = Y_{ij}/n_{ij} = \bar{Y}_{ij}, j = 1, \ldots, b_i; i = 1, \ldots, a. \) The g-inverse corresponding to this solution is verified to be \( G = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \) where \( D = \text{diag}(1/n_{ij}). \)
10.1. The form of $\text{Var}(\hat{\mu}_{\text{OLS}})$ follows from (10.1.6) using $\text{Var}(\hat{Y}_i) = \sigma^2 + \frac{\sigma^2}{n_i}$, and simplifying.

10.2. Recall that $N = na$, $1_N'(\sum_{i=1}^{a^+} J_n) = n1_N'$; $J_N'(\sum_{i=1}^{a^+} J_n) = nJ_N$; and $(\sum_{i=1}^{a^+} J_n)(\sum_{i=1}^{a^+} J_n) = n(\sum_{i=1}^{a^+} J_n)$. Write the model as $y = X\beta + e$, where $X = (x_0, \cdots, x_a)$, with $x_0 = 1_N$, and $x_i$ is an $N$-dimensional vector with 1 in the $i$th place and 0 elsewhere. Then, $Y_i = y'x_0$, $Y_i = y'x_i$, $i = 1, \cdots, a$, and

\[
\text{SSA} = \sum_{i=1}^{a} Y_i^2/n - NY^2 \\
= y'(\frac{1}{n} \sum_{i=1}^{a^+} x_i x'_i - \frac{1}{N} x_0 x_0')y \\
= y'(\frac{1}{n} \sum_{i=1}^{a^+} J_n - \frac{1}{N} J_N)y.
\]