7.3. We can write $H$ as $C'\beta = d$ with $C' = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$, and $d = (0, 0)'$. Then, $F(H) = (N - 6)(SSE_H - SSE)/2SSE \sim F_{2, N - 6}$ under $H$, where $SSE = y'(I_N - P) y$, $SSE_H = y'(I_N - P_H) y$, and $P_H = P - P_1 = X(X'X)^{-1}X' - X(X'X)^{-1}C(CX'X)^{-1}C_1(X'X)^{-1}X'$. If we replace $y$ by $y - \bar{y}_1 N$, then, $SSE_{new} = (y - \bar{y}_1 N)'(I_N - P)(y - \bar{y}_1 N) = SSE$, since $(I_N - P)1_N = 0$. A similar reasoning gives $SSE_{new,H} = SSE_H$, so that $F(H)$ is unaltered.

7.10. The two lines intersect at $(x_0, y_0)$ if and only if they intersect at a point $(0, y_0)$. This is seen by writing a shifted model $Y_{i,t} = \beta_{1,0} + \beta_{1,1}X_{i,t} + \epsilon_{i,t}$, where $\beta_{1,0} = \beta_{i,0} + \beta_{i,1}x_0$, and $X_{i,t} = (X_{i,t} - x_0)$. To test $\beta_{1,0} = \beta_{2,0} = \beta$, the $F$-statistic is

$$F(H) = (n_1 + n_2 - 4)(SSE_H - SSE)/SSE \sim F_{1, n_1 + n_2 - 4}$$

under $H$. Here, $SSE$ is the full model sum of squares given in (7.3.17), and

$$SSE_H = \sum_{i=1}^{2} \sum_{t=1}^{n_i} (Y_{i,t} - \hat{\beta}_H - \hat{\beta}_{1,1,H}X_{i,t})^2,$$

and

$$\hat{\beta}_H = \frac{\sum_{i=1}^{n_1} Y_{i,t} - \sum_{i=1}^{2} \sum_{t=1}^{n_i} \frac{X_{i,t} Y_{i,t}}{X_{i,t}^2}}{n_1 + n_2 - \sum_{i=1}^{2} \sum_{t=1}^{n_i} \frac{X_{i,t}^2}{X_{i,t}^2}},$$

where $X_i = \sum_{i} X_{i,t}$, and $\hat{\beta}_{1,1,H} = \frac{\sum_{i=1}^{n_i}(Y_{i,t} - \hat{\beta}_H)X_{i,t}}{\sum_{i} X_{i,t}^2}$, $i = 1, 2$. 

(a) If $c'\beta_1$ is estimable in the full model (7.4.1) with $c'\beta_1 = (c', 0) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, then $(c', 0) = t'(X_1, X_2)$ for some $t \in \mathbb{R}^n$, or $t'X_1 = c'$ and $t'X_2 = 0$.

\[ \implies t'X_1 = c' \]

\[ \implies c'\beta_1 \text{ is estimable in the reduced model (7.4.8).} \]

However, consider the situation in which $X_1$ is full column rank, so $\beta_1$ is estimable in the model (7.4.8) with $\beta_1 = (X_1'X_1)^{-1}X_1'Y$. However, if all column of $X_2$ are linear combination of column of $X_1$, then $\beta_1$ may not be estimable in the full model.

Example: One way ANOVA, $E(Y_{ij}) = \mu + \tau_i, j = 1, \ldots, n_i; i = 1, \ldots, k$ vs $E(Y_{ij}) = \tau_i$.

In the reduced model, all $\tau_i$'s are estimable, but in the full model, only contrasts are estimable.

(b) Using Result 7.4.2, and $c' = t'X_1$ since $c'\beta_1$ is estimable by Result 4.3.1

\[ E(c'\beta_{1,H}^0) = c'E(\beta_{1,H}^0) \]

\[ = c'H_1\beta_1 + c'(X_1'X_1)^{-1}X_1'X_2\beta_2 \]

\[ = t'X_1H_1\beta_1 + t'X_1(X_1'X_1)^{-1}X_1'X_2\beta_2 \]

\[ = t'X_1\beta_1 + t'P_1X_2\beta_2 \]

\[ = c'\beta_1 + t'P_1X_2\beta_2 \]

Then $t'P_1X_2\beta_2 \neq 0$ for underfitting $\implies c'\beta_{1,H}^0$ is a biased estimator.

(c) $E(c'\beta_1^0) = c'E(\beta_1^0) = c'H_1\beta_1$ by Result 7.4.3. Now using $c' = t'X_1$

\[ \implies E(c'\beta_1^0) = t'X_1H_1\beta_1 = t'X_1(X_1'X_1)^{-1}X_1'X_1\beta_1 = t'X_1\beta_1 = c'\beta_1 \]

which is unbiased.

i) for overfitting: $Y = (X_1, X_2)$

\[ c'\beta_1^0 = (c', 0)(X'X)^{-1}X'Y = t'X(X'X)^{-1}X'Y = t'PY \]

\[ Var(c'\beta_1^0) = t'P\sigma^2I P^t = \sigma^2 t'P \]

ii) for true model: $Y = X_1\beta_1 + \epsilon$

\[ c'\beta_{1,H}^0 = c'(X_1'X_1)^{-1}X_1'Y = t'X_1(X_1'X_1)^{-1}X_1'Y = t'P_1Y \]

\[ Var(c'\beta_{1,H}^0) = t'P_1\sigma^2I P_1t = \sigma^2 t'P_1t \]

We know $P = P_1 + P_2$ where $P_2 = P_1^\perp, P_1 \subset P$.

Since $t'P = t'P_1 + t'P_2$, $(t'P)(t'P)' \leq (t'P_1)(t'P_1)'$. Therefore,

\[ Var(c'\beta_1^0) - Var(c'\beta_{1,H}^0) = \sigma^2 t'P t - \sigma^2 t'P_1t = \sigma^2(t'P)(t'P)' - \sigma^2(t'P_1)(t'P_1)' \geq 0 \]

So $Var(c'\beta_1^0) > Var(c'\beta_{1,H}^0)$
7.15. The restricted least squares estimate of \( \beta \) subject to \( 1'\beta = 0 \) is \( \hat{\beta}_r = y - \hat{Y}_14 \), where \( \hat{Y} = \sum_{i=1}^4 Y_i/4 \). Also, \( SSE_r = 4\hat{Y}'^2 \). The reduced and restricted model consists of imposing \( H : \beta_1 = \beta_2 \), i.e.,
\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.
\]
We can show that (see section 7.5) \( \hat{\beta}_{r,H} = y - z \), where
\( z' = (\hat{Y} + \frac{1}{2} (Y_1 - Y_2), \hat{Y} - \frac{1}{2} (Y_1 - Y_2), \hat{Y}, \hat{Y}) \). Also, \( SSE_{r,H} = 4(\hat{Y})^2 + (Y_1 - Y_2)^2/2 \). Then,
\[
F_0 = \frac{2(Y_1 - Y_2)^2}{(\sum Y_i)^2}.
\]

7.18. \( Y_0 = x_0'\beta + \varepsilon_0 \), where \( \varepsilon_0 \sim N(0, \sigma^2) \). By the Gauss-Markov Theorem, \( \hat{Y}_0 = x_0'\hat{\beta} \sim N(x_0'\beta, \sigma^2 x_0'(X'X)^{-1}x_0) \).

(a) \( E(Y_0) = E(x_0'\beta + \varepsilon_0) = x_0'\beta \), and since \( \varepsilon_0 \) is independent of \( \varepsilon \),
\[
Var(Y_0 - \hat{Y}_0) = Var(Y_0) + Var(\hat{Y}_0) = \sigma^2 + \sigma^2 x_0'(X'X)^{-1}x_0.
\]
Replacing \( \sigma^2 \) by its least squares estimate gives the required result. We know that \( (N - p)\sigma^2/\sigma^2 \sim \chi^2_{N - p} \). The distribution of the given statistic is therefore \( t_{N - p} \).

7.23. (a) We can verify that the least squares solutions are \( \mu^0 = \bar{Y}, \tau^0 = Y_1 - \bar{Y}, \ldots, \tau^b = Y_4 - \bar{Y} \), so that \( SSE = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_i - \bar{Y}_j + \bar{Y})^2 + \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \bar{Y}_{ij})^2 \) with d.f. \( abn - (a + b - 1) = n^* \), say.
We have \( \hat{\sigma}^2 = SSE/n^* \). Let \( \mu_{ij} = \mu_i + \tau_j + \beta_{ij} \), with \( \mu_{ij} = \mu_i^0 + \tau_j^0 + \beta_{ij} \). The joint pdf of \( Y_{ijk} \) is
\[
(2\pi\sigma^2)^{-abn/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_i \sum_j \sum_k (Y_{ijk} - \mu_{ij})^2\right\}
= (2\pi\sigma^2)^{-abn/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_i \sum_j \sum_k (Y_{ijk} - \bar{\mu}_{ij})^2\right\}
+ n \sum_i \sum_j (\bar{\mu}_{ij} - \bar{\mu}_{ij})^2,
\]
because the product term vanishes by independence. From the properties of exponential families, it follows that \( \bar{\sigma}^2 \) and \( \bar{\mu}_{ij} \) are jointly complete, sufficient statistics for \( \sigma^2 \) and \( \mu_{ij} \). Since \( E(\bar{\sigma}^2) = \sigma^2 \), \( \bar{\sigma}^2 \) is unbiased and a function of the complete, sufficient statistics, so it must be the UMVUE of \( \sigma^2 \).

(b) Suppose \( X = (X_1, X_2, X_3) \), where \( X_1 \) corresponds to \( \mu \), \( X_2 \) corresponds to \( \tau_1, \ldots, \tau_a \), and \( X_3 \) corresponds to \( \beta_1, \ldots, \beta_b \). We know that \( M(X_1) \perp M(X_2) \), \( i \neq j \), i.e., the manifolds are orthogonal. Now, \( Z_1 = \sum_{i=1}^a c_i \tau_i^0 \in M(X_2) \) and \( Z_2 = \sum_{j=1}^b c_j \beta_j^0 \in M(X_3) \), so that \( Z_1 \) and \( Z_2 \) are independent. Let \( c \) denote an \( N = 1 + a + b \) dimensional vector with \( c_1, \ldots, c_a \) in positions \( 2, \ldots, a + 1 \), and zeroes elsewhere. Since \( Z_1 = c'(X'X)^{-1}X'y \), and \( c'(X'X)^{-1}X'[I - X(X'X)^{-1}X'] = 0 \), it follows that \( Z_1 \) and \( SSE \) are independent, i.e., \( Z_1 \) and \( U \) are independent. A similar calculation shows that \( Z_1 \) and \( U \) are independent.
7.29. Consider the nested sequence of subspaces $S_{H_0} \supset S_{H_1} \supset S_{H_2} \supset S_{H_3} \supset S_{H_4} \supset \{0\}$. The least squares estimates are: $\mu_0 = \overline{Y}$, $\tau_i^0 = \overline{Y}_i - \overline{Y}$, $\beta_j^0 = \overline{Y}_{.j} - \overline{Y}$, $\gamma_k^0 = \overline{Y}_{.k} - \overline{Y}$, $(\tau\beta)_i^0 = \overline{Y}_{ij} - \overline{Y}$, and $(\beta\gamma)_jk^0 = \overline{Y}_{ijk} - \overline{Y}$. Hence, $Q(H_0) = \sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij} - \overline{Y}_{.jk} + \overline{Y}_{.j})^2$ with $b(a + c - 1)$ d.f., $SSE = \sum_i \sum_j \sum_k (Y_{ijk} - \overline{Y}_{ij} - \overline{Y}_{.jk} + \overline{Y}_{.j})^2$ with $b(a - 1)(c - 1)$ d.f., $Q(H_0|H_1) = a \sum_j \sum_k (Y_{.jk} - \overline{Y}_{.j} - \overline{Y}_{..} + \overline{Y}_{..})^2$ with $(b - 1)(c - 1)$ d.f., $Q(H_2|H_3) = c \sum_i \sum_k (Y_{.ik} - \overline{Y}_{.i} - \overline{Y}_{.j} + \overline{Y}_{..})^2$ with $(a - 1)(b - 1)$ d.f., $Q(H_4|H_5) = abc \sum_i (Y_{.i} - \overline{Y}_{..})^2$ with $(a - 1)$ d.f., $Q(H_5|H_6) = abc \sum_i \overline{Y}_{.i}$ with 1 d.f. The ANOVA table can be completed based on this information.