Question 1

(10 x 4 = 40 points)

**Special Instructions for this question:** You DO NOT have to show work for (a) through (e). Write the correct choice (A/B/C/D) inside the box on the right hand side. Any scratch work will not be evaluated, only the answer in the box matters!

(a) Suppose that $A$ and $B$ are independent events with $P(A) = .40$ and $P(B) = .70$, then $P(B' \mid A')$ is

A and B are independent implies $P(B' \mid A') = P(B')$

Answer: .30

(b) The sample space for the experiment in which a six-sided die is thrown twice (order important) consists of how many outcomes?

Order is important, each draw has 6 outcomes. Now use product rule (6X6).

Answer: 36

(c) If $P(A) = .30$, $P(B) = .60$, and $P(A \cap B) = .18$, then events $A$ and $B$ are

Answer: independent

(d) How many ordered rearrangements of size 3 can be constructed from the set (A, B, C, D, E)?

$5 \times 4 \times 3 = P_{3,5}$

Answer: 60

(e) Which of the following is (are) true if events $A$ and $B$ are mutually exclusive with $P(B) > 0$?

A. $P(A \cap B) = 0$
B. $P(A \mid B) = 0$
C. $A$ and $B$ cannot be independent

All these three are valid.

Answer: E.

(f) If $P(A \mid B) = P(A)$, and $P(B \mid A) = P(B)$, then events $A$ and $B$ are said to be

Answer: independent
(g) Let $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event $B$,

$$P(B) = P(B | A_1)P(A_1) + \cdots + P(B | A_k)P(A_k)$$

is well known as

**Answer:** The law of total probability

(h) In many counting and probability problems, a useful configuration that can be used to represent pictorially all the possibilities (sample space and probabilities) is the

**Answer:** Tree diagram

(i) A task consists of three stages. There are five ways to accomplish the first stage, four ways to accomplish the second stage, and three ways to accomplish the third stage. The total number of ways to accomplish the task is

**Answer:** 60

(j) For any two events $A$ and $B$,

**Answer:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
Question 2  

(15 points)

A small manufacturing company will start operating a night shift. There are 12 [or 15] machinists employed by the company. A night crew consists of 3 machinists selected at random.

(a) (2 point) How many different night crews are possible?

Out of 12, \( \binom{12}{3} = \frac{12!}{3! \times 9!} = \frac{12 	imes 11 	imes 10}{3 	imes 2 \times 1} = 220 \) or

Out of 15, \( \binom{15}{3} = 455 \)

(b) (3 points) If the machinists are ranked 1, 2, \ldots, 12 in order of competence, how many of these crews would not have the best machinist?

The best machinist is the one with rank 1 who is not to be included. So, choose 3 from the rest (12-1) = 11 or (15-1) = 14, i.e., Select three by keeping the best out of the selection process. That equals \( \binom{11}{3} = \frac{11!}{3! \times 8!} = \frac{11 	imes 10 	imes 9}{3 	imes 2 \times 1} = 165 \) or

\( \binom{14}{3} = 364 \)

(c) (3 points) How many of the crews would have at least 1 of the 7 best machinists?

At least 1 form the first 7 would require a lot more calculations. Since we already have (a) we can first calculate the number of crews which have none of the 7 best machinists i.e., all the 3 crew members are from those ranked 8, \ldots, 12: \( \binom{5}{3} = 10 \).

So, the answer is 220 - 10 = 210.

Of course, you can still compute the long way: “exactly 1 from the first 7” U “exactly 2 from the first 7” U “exactly 3 from the first 7”: \( \binom{7}{1} \binom{5}{2} + \binom{7}{2} \binom{5}{1} + \binom{7}{3} \binom{5}{0} = 210 \).

OR for 15 machinists in all, we have top 7 and remaining 8. So none of best 7 would equal \( \binom{8}{3} \binom{5}{0} = 56 \), and the answer would be 455 – 56 = 399.

(d) (3 points) If one of these crews of three person is selected at random to work on a particular night, what is the probability that the best machinist will not work that night?

From parts (a) and (b), \( P(\text{the best machinist will not work that night}) = \frac{165}{220} = \frac{3}{4} = 0.75 \)

Similarly, for 15 machinists, this probability would be \( \frac{364}{455} = 0.8 \)

(e) (4 points) Assuming that five machinists hail from east coast and all the rest from west coast, find the probability that the selected crew will have at most one from the east coast.

Let the event \( A = \{ \text{at most one member from East Coast} \} \). Then

\# of crews with at most one member from East Coast = \( N(A) = \# \) of crews with (3 from West Coast and 0 from East Coast) + \# of crews with (2 from West Coast and 1 from East Coast) = \( \binom{7}{3} \binom{5}{0} + \binom{7}{2} \binom{5}{1} = 140 \). Therefore, \( P(A) = \frac{140}{220} = 0.6363 \).

Similarly, for 15 machinists, we have five from East Coast and 10 from West coast. Thus \( N(A) = \binom{10}{3} \binom{5}{0} + \binom{10}{2} \binom{5}{1} = 345 \), and \( P(A) = \frac{345}{455} = 0.7582 \)
Question 3  

A large company offers its employees two different health insurance plans and two different dental insurance plans. Plan 1 of each type is relatively inexpensive, but restricts the choice of providers, whereas plan 2 is more expensive but more flexible. The accompanying table gives the percentages of employees who have chosen the various plans:

<table>
<thead>
<tr>
<th>Health Plan</th>
<th>D1</th>
<th>D2</th>
<th>Col. Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>27%</td>
<td>14%</td>
<td>41%</td>
</tr>
<tr>
<td>H2</td>
<td>24%</td>
<td>35%</td>
<td>59%</td>
</tr>
<tr>
<td>Row Sum</td>
<td>51%</td>
<td>49%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Suppose that an employee is randomly selected and both the health plan and dental plan chosen by the selected employee are determined.

Let H1 and H2 represent the two health plans. Let D1 and D2 represent the two dental plans.

a. (3 points) List the four simple events.

\{H1, D1\}, \{H1, D2\}, \{H2, D1\}, \{H2, D2\}

b. (4 points) Calculate the probability that the selected employee has chosen the more restrictive plan of each type.

\[ P\{H1, D1\} = 0.27 \]

c. (4 points) Calculate the probability that the employee has chosen the more flexible dental plan?

\[ P\{D2\} = P\{H1, D2\}, \{H2, D2\} = 0.14 + 0.35 = 0.49 \]

d. (4 points) Which health plan is popular in this group. Explain.

\[ H_2, \text{since } P(H_2) > 0.5 \]
Question 4 (20 points)

At a certain gas station selling three types of unleaded gas, 40% of the customers buy regular gas \((A_1)\), 35% buy plus gas \((A_2)\), and 25% buy premium gas \((A_3)\). Of those customers buying regular gas, only 30% fill their tanks (event \(B\)). Of those customers buying plus, 60% fill their tanks, and of those using premium, 50% fill their tanks.

\[
P(A_1) = .40, \quad P(A_2) = .35, \quad P(A_3) = .25; \quad P(B|A_1) = .30, \quad P(B|A_2) = .60, \quad P(B|A_3) = .50.
\]

Therefore,

\[
P(A_1 \cap B) = P(A_1) \cdot P(B|A_1) = (.40)(.30) = .12
\]

\[
P(A_2 \cap B) = P(A_2) \cdot P(B|A_2) = (.35)(.60) = .21
\]

\[
P(A_3 \cap B) = P(A_3) \cdot P(B|A_3) = (.25)(.50) = .125
\]

(a) (5 points) What is the probability that the next customer will buy premium unleaded gas and fill the tank? or buy premium unleaded gas and not fill the tank?

\[
P(\text{buy premium gas and fill the tank}) = P(A_3 \cap B) = P(A_3) \cdot P(B|A_3) = (.25)(.50) = .125
\]

\[
P(\text{buy premium gas and not fill the tank}) = P(A_3 \cap B') = P(A_3) \cdot P(B'|A_3) = (.25)(.50) = .125
\]

(b) (5 points) What is the probability that the next customer fills the tank? or doesn’t fill the tank?

\[
P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) = .12 + .21 + .125 = .455
\]

Therefore, \(P(B') = 1 - P(B) = .545\)

or \(P(B') = P(A_1 \cap B') + P(A_2 \cap B') + P(A_3 \cap B') = .28 + .14 + .125 = .545\)

(c) (7 points) Given that the next customer fills the tank [or doesn’t fill the tank], find the conditional probabilities of the event that

(i) Regular gas is purchased.

\[
P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.12}{.455} = .264 \quad \text{or} \quad P(A_1|B') = \frac{P(A_1 \cap B')}{P(B')} = \frac{.28}{.545} = .5138
\]

(ii) Plus gas is purchased.

\[
P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{.21}{.455} = .462 \quad \text{or} \quad P(A_2|B') = \frac{P(A_2 \cap B')}{P(B')} = \frac{.14}{.545} = .2569
\]

(iii) Premium gas is purchased.

\[
P(A_3|B) = \frac{P(A_3 \cap B)}{P(B)} = \frac{.125}{.455} = .275 \quad \text{or} \quad P(A_3|B') = \frac{P(A_3 \cap B')}{P(B')} = \frac{.125}{.545} = .2293
\]

d) (3 points) Are the events \(B\) and \(A_1\) independent? Explain.

Since \(P(B)\) is not equal to \(P(B|A_1)\), these events are not independent.