Question 1

Proof: \( X_1, \ldots, X_n \overset{iid}{\sim} U(\theta - \frac{1}{2}, \theta + \frac{1}{2}) \).

Let \( Y_i = X_i - \theta + \frac{1}{2}, i = 1, \ldots, n \), then \( Y_1, \ldots, Y_n \overset{iid}{\sim} U(0, 1) \) and 
\( Y_{(1)} = X_{(1)} - \theta + \frac{1}{2}, Y_{(n)} = X_{(n)} - \theta + \frac{1}{2} \).

It’s known that \( Y_{(1)} \sim Beta(1, n) \) and \( Y_{(n)} \sim Beta(n, 1) \). Thus
\[
E(Y_{(1)}) = \frac{1}{n+1} \quad E(Y_{(n)}) = \frac{n}{n+1}
\]
and
\[
E(X_{(n)} - X_{(1)}) = E(Y_{(n)} - Y_{(1)}) = \frac{n-1}{n+1}
\]

i.e. \( E\left( X_{(n)} - X_{(1)} - \frac{n-1}{n+1} \right) = 0 \)

Let 
\( h(I) = X_{(n)} - X_{(1)} - \frac{n-1}{n+1} \)

, then the distribution of \( X_{(n)} - X_{(1)} \) does not depend on \( \theta \).

\( E(h(I)) = 0 \quad \forall \ \theta \)

However, \( P(h(I) = 0) \neq 1 \), so \( T(X) = (X_{(1)}, X_{(n)}) \) is not complete.

Question 2

\( f(x, y) \) has continuous partial derivatives of the first and second order on \( \mathbb{R}^2 \).

\[
\nabla f(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \quad H(f(x, y)) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
\]

\( \det(H(f(x, y))) = 4 > 0 \) and the matrix is diagonal with positive diagonal elements. Thus \( H(f(x, y)) \) is positive definite and \( f(x, y) \) is convex. At point \( (1, 0) \), the support hyperplane \( L(X) \) is

\[
L(X) = f(1, 0) + \nabla f(1, 0) (x, y) = f(1, 0) + 2(x - 1, y) = 2x - 1
\]
Question 3

Proof: First show \(-l(w)\) is convex.

\[
-l(w) = \sum_{i=1}^{n} \log[1 + \exp(-y_i w^T x_i)]
\]

\[
\nabla(-l(w)) = \sum_{i=1}^{n} \begin{bmatrix}
-\frac{y_i x_{i1} \exp(-y_i w^T x_i)}{1 + \exp(-y_i w^T x_i)} \\
\vdots \\
-\frac{y_i x_{ik} \exp(-y_i w^T x_i)}{1 + \exp(-y_i w^T x_i)}
\end{bmatrix}
\]

\[
H(-l(w)) = \sum_{i=1}^{n} \begin{bmatrix}
\frac{(y_i x_{i1})^2 \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} & \cdots & \frac{(y_i x_{i1} x_{ik}) \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} \\
\vdots & \ddots & \vdots \\
\frac{(y_i x_{ik})^2 \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} & \cdots & \frac{(y_i x_{ik})^2 \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2}
\end{bmatrix}
\]

\[
= \sum_{i=1}^{n} \frac{\exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} x_i x_i^T
\]

\(H(-l(w))\) is positive definite since \(x_i x_i^T\) is positive definite and hence \(-l(w)\) is strictly convex.

Let

\[
\begin{align*}
f(w_0) &= -l(w_0) = -m \\
\end{align*}
\]

be the minimum and \(w_0\) denote MLE. Suppose \(\exists w_1 \text{ s.t.}\)

\[
\begin{align*}
f(w_1) &= f(w_0) = -m \quad \text{and} \quad w_1 \neq w_0
\end{align*}
\]

Then \(\forall 0 \leq r \leq 1,\)

\[
rf(w_0) + (1-r)f(w_1) > f(rw_0 + (1-r)w_1)
\]

by convexity of \(-l(w)\). In other words,

\[
-m > f(rw_0 + (1-r)w_1)
\]

This is an contradiction to the fact that \(-m\) is the minimum of \(f(w)\).

Thus MLE is unique.
**Question 4**

Proof:

\[ E_F[h(x)] = \int h(x) \, dF \]

\[ = \int \int_0^{h(x)} dt \, dF \]

\[ = \int_0^{h(x)} \int_{\{x \in \mathbb{R}^k : h(x) > t\}} dF \, dt \]

\[ = \int_0^{\infty} F(h(x) > t) \, dt \]

Similarly

\[ E_G[h(x)] = \int_0^{\infty} G(h(x) > t) \, dt \]

We have

\[ E_G[h(x)] - E_F[h(x)] \]

\[ = \int_0^{\infty} [G(h(x) > t) - F(h(x) > t)] \, dt \]

\[ = \int_0^{\infty} [1 - G(h(x) < t)] - [1 - F(h(x) < t)] \, dt \]

\[ = \int_0^{\infty} [F(h(x) < t) - G(h(x) < t)] \, dt > 0 \quad (*) \]

Notice that the set \( H = \{ x : h(x) \leq t \} \) is a convex set. This is because

\( \forall x, y \in H \)

\( h(rx - (1-r)y) \leq rh(x) + (1-r)h(y) \leq t \)

So \( rx + (1-r)y \in H \).

Thus, (*) implies there is a convex set \( A \in \mathbb{R}^k \), with \( 0 \in A \) s.t. \( \forall t_0 \in A \),

\( F(h(x) \leq t_0) - G(h(x) \leq t_0) \geq 0 \)

\( \Rightarrow F(A) \geq G(A) \) for such \( A \)

**Question 5**
∇f(x, y) = \begin{pmatrix} -\alpha x^{\alpha - 1} y^{1 - \alpha} \\ -x^\alpha (1 - \alpha) y^{-\alpha} \end{pmatrix}

H(f(x, y)) = \begin{pmatrix} -\alpha(\alpha - 1) x^{\alpha - 2} y^{1 - \alpha} & -\alpha(1 -\alpha) x^{\alpha - 1} y^{-\alpha} \\ -\alpha(1 - \alpha) x^{\alpha - 1} y^{-\alpha} & \alpha(1 - \alpha) x^\alpha y^{\alpha - \alpha - 1} \end{pmatrix}

\det(H) = \alpha^2(1 - \alpha)^2 x^{2\alpha - 2} y^{2\alpha} - \alpha(1 - \alpha)^2 x^{2\alpha - 2} y^{2\alpha} = 0

tr(H) = \alpha(1 - \alpha) x^{\alpha - 2} y^{-\alpha - 1}[x^2 + y^2] > 0

The eigenvalue λ's are the roots of the equation

\[ \lambda^2 - \lambda tr(H) + det(H) = 0 \]

\[ \Rightarrow \lambda(\lambda - tr(H)) = 0 \]

\[ \Rightarrow \lambda = 0 \quad \text{or} \quad \lambda = tr(H) > 0 \]

Both of the roots are non-negative and hence the Hessian matrix is positive semidefinite.

\[ \Rightarrow f(x, y) = -x^\alpha y^{1 - \alpha} \quad \forall \ 0 < \alpha < 1 \]

is convex on \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.

(b) Use Jensen’s Inequality

\[ E(f(x)) \geq f(E(x)) \]

for convex f(\cdot). Here we have

\[ f(x, y) = -x^\alpha y^{1 - \alpha} \quad \forall \ 0 < \alpha < 1 \]

So

\[ E[-x^\alpha y^{1 - \alpha}] \geq -(EX)^\alpha (EY)^{1 - \alpha} \]

i.e.

\[ E(X^\alpha Y^{1 - \alpha}) \leq (EX)^\alpha (EY)^{1 - \alpha} \]
(c) We first show that the natural parameter space

$$\Theta = \{ \eta : \int \exp(\eta' T) d\mu < \infty \}$$

is convex.

Suppose $$\eta_1, \eta_2 \in \Theta$$ and for $$0 \leq r \leq 1$$

$$\int \exp(r \eta_1' T + (1 - r) \eta_2' T) \, d\mu$$

$$\propto E[ (e^{\eta_1' T})^r (e^{\eta_2' T})^{1-r} ]$$

$$\leq \left[ E(e^{\eta_1' T})^r (E e^{\eta_2' T})^{1-r} \right]$$

$$\propto \left( \int \exp(\eta_1' T) \, d\mu \right)^r \left( \int \exp(\eta_2' T) \, d\mu \right)^{1-r}$$

$$\leq \infty$$ since $$\eta_1, \eta_2 \in \Theta$$

Therefore $$r \eta_1 + (1 - r) \eta_2 \in \Theta$$ and hence $$\Theta$$ is convex.

$$A(\eta)$$ is defined on a convex set $$\Theta$$ since $$\text{Cov}(T) = A(\eta)$$ and $$\text{Cov}(T)$$ is positive semidefinite. Thus $$A(\eta)$$ is a convex function on $$\Theta$$. 

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