Question 1

(a) 
\[ f(x) = \prod_{i=1}^{n} [\theta x_i^{\theta-1} I_{(0,1)}(x_i)] \]
\[ = \theta^n \left( \prod_{i=1}^{n} x_i \right)^{\theta-1} \prod_{i=1}^{n} I_{(0,1)}(x_i) \]
\[ = \left[ I_{(0,1)}(\min\{x_1, \ldots, x_n\}) I_{(0,1)}(\max\{x_1, \ldots, x_n\}) \right] \left[ \left( \prod_{i=1}^{n} x_i \right)^{\theta-1} \theta^n \right] \]

By Neyman’s factorization theorem, \( T = \prod_{i=1}^{n} x_i \) is the sufficient statistics for \( \theta \).

(b) 
\[ f(x) = \prod_{i=1}^{n} \theta a x_i^{a-1} \exp\{-\theta x_i^a\} I_{(0,1)}(x_i) \]
\[ = (\theta a)^n \left( \prod_{i=1}^{n} x_i \right)^{a-1} \exp\{-\theta \sum_{i=1}^{n} x_i^a\} I_{(0,1)}(\min\{x_1, \ldots, x_n\}) \]
\[ = \left[ a^n \left( \prod_{i=1}^{n} x_i \right)^{a-1} I_{(0,1)}(\min\{x_1, \ldots, x_n\}) \right] \left[ \theta^n \exp\{-\theta \sum_{i=1}^{n} x_i^a\} \right] \]

By Neyman’s factorization theorem, \( T = \sum_{i=1}^{n} x_i^a \) is the sufficient statistics for \( \theta \).

(c) 
\[ f(x) = \prod_{i=1}^{n} \frac{\theta a^\theta}{x_i^{\theta+1}} I_{(a,\infty)}(x_i) \]
\[ = \frac{\theta^n a^\theta}{\left( \prod_{i=1}^{n} x_i \right)^{\theta+1}} I_{(a,\infty)}(\min\{x_1, \ldots, x_n\}) \]
\[ = \left[ I_{(a,\infty)}(\min\{x_1, \ldots, x_n\}) \right] \left[ \frac{\theta^n a^\theta}{\left( \prod_{i=1}^{n} x_i \right)^{\theta+1}} \right] \]
By Neyman’s factorization theorem, $T = \prod_{i=1}^{n} x_i$ is the sufficient statistics for $\theta$.

**Question 2**

(a) For any $\theta \in \Theta$, the posterior distribution of $\theta|x$ is:

$$f(x|\theta)\pi(\theta) \over \sum_{\theta_i \in \Theta} f(x|\theta_i)\pi(\theta_i)$$

where $\pi(\theta)$ is any prior on $\theta$

By the condition, we know it equals to some function $g(\theta, T(x))$, i.e.

$$f(x|\theta)\pi(\theta) \over \sum_{\theta_i \in \Theta} f(x|\theta_i)\pi(\theta_i) = g(\theta, T(x))$$

where $g(x, T(x))$ is a function of $\theta$ and $T(x)$ only. Thus

$$f(x|\theta) = g(x, T(x)) \over \pi(\theta) \sum_{\theta_i \in \Theta} f(x|\theta_i)\pi(\theta_i)$$

By factorization theorem, $T(x)$ is sufficient for $\theta$.

(b) If $T(x)$ is sufficient, then $f(x|\theta)$ can be written as

$$f(x|\theta) = g(\theta, T(x))h(x)$$

Let $\pi(\theta)$ be an arbitrary prior distribution, then the posterior of $\theta$ is

$$f(x|\theta)\pi(\theta) \over \sum_{\theta_i \in \Theta} f(x|\theta_i)\pi(\theta_i) = g(\theta, T(x)) \over \sum_{\theta_i \in \Theta} g(\theta_i, T(x))\pi(\theta_i) \pi(\theta)$$

The posterior depends on $x$ only through $T(x)$. By factorization theorem, $T(x)$ is sufficient for $\theta$.

**Question 3  (Problem 6.7)**

(a) $\theta = \{\xi, \eta, \sigma, \tau : -\infty < \xi, \eta < \infty, 0 < \sigma, \tau\}$ The parameter space
contains 4-dimensional rectangle space.

\[ f(x) = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} (x_i - \xi)^2} \]

\[ = \frac{1}{(2\pi)^{m/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum x_i^2 + \frac{\xi}{\sigma^2} \sum x_i - \frac{m\xi^2}{2\sigma^2} - m \log \sigma \right\} \]

\[ f(y) = \frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2\tau^2} \sum y_i^2 + \frac{\eta}{\tau^2} \sum y_i - \frac{n\eta^2}{2\tau^2} - n \log \tau \right\} \]

Thus, \((x,y)\) is distributed as \(f(x)f(y)\) with parameter space \(\theta = \{\xi, \eta, \sigma, \tau: -\infty < \xi, \eta < \infty, 0 < \sigma, \tau\}\). The sufficient statistics \(T = \{\sum x_i, \sum x_i^2, \sum y_i, \sum y_i^2\}\) is also minimal since \((x,y)\) belongs to exponential family.

(b) If \(\sigma = \tau\), then the parameter space becomes

\[ \Theta = \{\xi, \eta, \sigma : -\infty < \xi, \eta < \infty, 0 < \sigma < \infty\} \]

which contains a 3-dimensional triangle in it. The joint distribution of \((x,y)\) is

\[ f(x, y) = \frac{1}{(2\pi)^{(m+n)/2}} \exp \left\{ -\frac{1}{2\sigma^2} \left(\sum x_i^2 + \sum y_i^2\right) + \frac{\xi}{\sigma^2} \sum x_i + \frac{\eta}{\sigma^2} \sum y_i - \frac{m\xi^2 + n\eta^2}{2\sigma^2} - (m + n) \log \sigma \right\} \]

with natural parameter

\[ \{\eta_1, \eta_2, \eta_3 : \eta_1 = \frac{1}{2\sigma^2}, \eta_2 = \frac{\xi}{\sigma^2}, \eta_3 = \frac{\eta}{\sigma^2}\} \]

This distribution belongs to exponential family with full rank. By theorem 6.22, \(\{\sum x_i^2 + \sum y_i^2, \sum x_i, \sum y_i\}\) is complete.

(c) \(\xi = \eta\) and \(\xi, \sigma, \eta\) are arbitrary. The joint distribution of \((x,y)\) is

\[ f(x, y) = \frac{1}{(2\pi)^{(m+n)/2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum x_i^2 - \frac{1}{2\tau^2} \sum y_i^2 + \frac{\xi}{\sigma^2} \sum x_i + \frac{\eta}{\tau^2} \sum y_i - \frac{m\xi^2 + n\eta^2}{2\sigma^2} - m \log \sigma - \frac{n\eta^2}{2\tau^2} - n \log \tau \right\} \]

This belongs to exponential family which is full rank. Therefore, the minimal sufficient statistics is \(\{\sum x_i, \sum x_i^2, \sum y_i, \sum y_i^2\}\).
To show that the parameter space is full rank, let
\[ \eta_1 = -\frac{1}{2\sigma^2}, \eta_2 = -\frac{1}{2\tau^2}, \eta_3 = \frac{\xi}{\sigma^2}, \eta_4 = \frac{\eta_1}{\eta_2} \]

The natural parameter space is
\[ \Theta = \{(\eta_1, \eta_2, \eta_3, \eta_4) : \eta_1 < 0, \eta_2 < 0, \frac{\eta_3}{\eta_4} = \frac{\eta_1}{\eta_2} \} \]

If we pick
\[ \eta^{(1)} = (-1/2, -1/2, 1, 1) \]
\[ \eta^{(2)} = (-1/4, -1/4, 2, 2) \]
\[ \eta^{(3)} = (-1/9, -1/3, 3, 9) \]
\[ \eta^{(4)} = (-1/16, -1/16, 4, 16) \]
\[ \eta^{(5)} = (-1/25, -1/25, 5, 5) \]

then the 4 × 4 matrix
\[
\begin{vmatrix}
\eta^{(2)} - \eta^{(1)} \\
\eta^{(3)} - \eta^{(1)} \\
\eta^{(4)} - \eta^{(1)} \\
\eta^{(5)} - \eta^{(1)} \\
\end{vmatrix}
\neq 0
\]

Thus it’s full rank.

**Question 4  (Problem 6.20)**

(a)
\[
p_{\theta}(x) = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{1}{2\theta} \left( \sum x_i^2 - 2\theta \sum x_i + n\theta^2 \right) \right\}
= \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{1}{\theta} \left( \frac{1}{2} \sum x_i^2 \right) - \sum x_i + \frac{n}{2} \theta \right\}
\]

\[ T_1(x) = (\sum x_i, \sum x_i^2) \] is sufficient. Consider \( T_2(x) = \sum x_i^2 \) which is sufficient statistic also, but for \( x \neq y \) \( \exists T_2(x) = T_2(y) \) s.t. \( T_1(x) \neq T_1(y) \). Thus \( T_1(x) \) is not minimal.
To show \((\sum x_i, \sum x_i^2, \sum x_i^3)\) is minimal sufficient, we need to show that
\[\exists \eta^{(j)} = (\eta_1^{(j)}, \eta_2^{(j)}, \eta_3^{(j)}), \quad j = 0, 1, 2, 3\]
such that
\[
\begin{pmatrix}
    p(x, \eta^{(1)}) & p(x, \eta^{(2)}) & p(x, \eta^{(3)}) \\
    p(x, \eta^{(0)}) & p(x, \eta^{(0)}) & p(x, \eta^{(0)})
\end{pmatrix}
\]
is a one-to-one transformation of \(T = (\sum x_i, \sum x_i^2, \sum x_i^3)\).

From the pdf:
\[
C \exp(-n\theta^4) \exp\left(4\theta^3 \sum x_i - 6\theta^2 \sum x_i^2 + 4\theta \sum x_i^3 - \sum x_i^4\right)
\]
we know the natural parameters are
\[4\theta^3 = \eta_1, -6\theta^2 = \eta_2, 4\theta = \eta_3\]
and
\[
\left(\sum_{i=1}^3 (\eta_i^{(1)} - \eta_i^{(0)})T_i(x), \sum_{i=1}^3 (\eta_i^{(2)} - \eta_i^{(0)})T_i(x), \sum_{i=1}^3 (\eta_i^{(3)} - \eta_i^{(0)})T_i(x)\right)
\]
is minimal sufficient for any \(\eta^{(0)}, \eta^{(1)}, \eta^{(2)}, \eta^{(3)} \in \Theta\).

It can also be written as
\[
\begin{pmatrix}
    \eta_1^{(1)} - \eta_1^{(0)} & \eta_2^{(1)} - \eta_2^{(0)} & \eta_3^{(1)} - \eta_3^{(0)} \\
    \eta_1^{(2)} - \eta_1^{(0)} & \eta_2^{(2)} - \eta_2^{(0)} & \eta_3^{(2)} - \eta_3^{(0)} \\
    \eta_1^{(3)} - \eta_1^{(0)} & \eta_2^{(3)} - \eta_2^{(0)} & \eta_3^{(3)} - \eta_3^{(0)}
\end{pmatrix}
\begin{pmatrix}
    T_1 \\
    T_2 \\
    T_3
\end{pmatrix}
\]

Let
\[\eta^{(0)} = (0, 0, 0), \quad \eta^{(1)} = (1/2, -3/2, 2), \quad \eta^{(2)} = (4, -6, 4), \quad \eta^{(3)} = (32, -24, 8)\]
then it’s easy to see that \((\eta^{(1)} - \eta^{(0)}, \eta^{(2)} - \eta^{(0)}, \eta^{(3)} - \eta^{(0)})^t\) span \(E_3\)
and the determinant is \(72 \neq 0\). Thus, \((T_1, T_2, T_3)\) is minimal sufficient.

**Question 5**  (Problem 6.21)
Proof: \[ T = (\sum x_i, \sum x_i^2) \]

\[
E \left( \frac{\sum x_i}{n} \right) = E(\bar{x}) = Var(\bar{x}) + [E(\bar{x})]^2 = \frac{\xi^2}{n} + \xi^2 = \frac{n+1}{n} \xi^2
\]

\[
\Rightarrow E \left[ \frac{n}{n+1} \left( \frac{\sum x_i}{n} \right)^2 \right] = \xi^2
\]

\[
E \left( \sum x_i^2 \right) = nE(x_i^2) = 2n\xi^2 \Rightarrow E \left( \frac{\sum x_i^2}{2n} \right) = \xi^2
\]

\[
\Rightarrow E f(T(x)) = 0 \quad \text{where} \quad f(T(x)) = \frac{n}{n+1} \left( \frac{\sum x_i}{n} \right)^2 - \frac{\sum x_i^2}{2n}
\]

But \( f(T) \neq 0 \ a.e. \), thus \( T = (\sum x_i, \sum x_i^2) \) is not complete.