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Thurstone scaling in order statistics

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Abstract

Thurstone scaling is widely used for presenting the priorities among the compared items. The mean values of the quantiles corresponding to frequencies of each stimilus' preference over the other stimuli define the items' locations on the psychological continuum of the Thurstone scale. This paper considers an extension of the scale levels to the aggregates of the independent covariates. In a sense, it is similar to a multiple regression extension of the mean value of the dependent variable to its conditional mean expressed by the linear aggregate of the independent variables. A maximum likelihood objective constructed by the probabilities of the order statistics applied to the ranked or paired comparison data is suggested. Probit, logit and multinomial links are tried to obtain the Thurstonian scale exposition by the covariates, and to estimate probabilities of the items' choice. This approach is very convenient and can substantially enrich both theoretical interpretation and practical application of Thurstone modelling, particularly in marketing and advertising research.

Keywords: Ranks; Paired comparisons; Thurstone scale; Order statistics; Probit; Logit; Multinomial

1. Introduction

Thurstone scaling is the well-known tool for the estimation of preferences among objects by the observed frequencies of their paired comparisons [38–40]. The positioning of items on this scale can be found by averaging the percentiles of the standard normal distribution corresponding to the proportions of the respondents preferring one item over each of the others. This scaling is widely used in applied psychology, particularly, in marketing and advertising research [11,41,3,15,8]. Statistical approaches to the Thurstone scaling were considered by Mosteller [33] and Daniels [9], and various modifications of this model were developed in [34,14,10,36,12,30,21–23]. An approach closely related to Thurstone scaling and commonly used in educational testing is the IRT, or item response theory [16, 20,4]. It is also known in terms of latent trait factor analysis and is close to the Rasch model [1,2,35]. Comparisons between TS and IRT are given in [42,13]. For a one-parameter model of item discrimination with a constant parameter across the examenee and without the intercept parameter of item difficulty, the IRT model reduces to Thurstone scaling. A comprehensive review [43] considers the problems of construct theory in market and marketing researches and compares the measures and scales, particularly Thurstone and Likert scales, oriented to comparison across the items or the respondents useful possibility of constructing Thurstone scales via Maximum Likelihood (ML) objective

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applied directly to the individually elicited paired comparison data is considered in [24]. Such techniques were used in competitive sport judging/scoring systems [37,32,17]. The ML estimation yields a binary regression model with the coefficients corresponding to the items' positions on a Thurstone scale. In the current paper, we apply the ML approach to the probability density functions (p.d.f.) of the order statistics [19]. Using the marginal order statistics p.d.f. instead of a regular probability in the ML objective allows construction of a Thurstone scale not only for the paired comparisons, but for ranked data as well. In distinction to the techniques of ordinal regressions modelling when the differences of the cumulative probability functions are used with the ordered inputs [1,27,20,26] we employ the order statistics marginal p.d.f. corresponding specifically to each rank. This approach opens a promising possibility of presenting the Thurstone levels as aggregates of the independent covariates. In a sense, it is similar to a multiple regression extension of the mean value of the dependent variable to its conditional mean expressed by linear aggregate of the independent variables.

Thurstone scaling is based on the normal distribution assumption. Alternatively, the Bradley–Terry model for paired comparison and the Bradley–Terry–Luce extension to multiple comparisons model are known for the preference modelling [5,6,28,29,10,22,24]. We try the probit, logit and multinomial links to obtain the Thurstonian scales extended by the covariates, and to estimate probabilities of the items' choice. The suggested technique is very convenient and useful both for theoretical study and practical usage of Thurstone modelling in applied psychology, particularly, in marketing and advertising research.

This paper is organized as follows. Section 2 describes Thurstone scale in the Least Squares and in regular Maximum Likelihood approach. Section 3 introduces Thurstone Scaling via the order statistics in Maximum Likelihood objective. Section 4 presents numerical examples from marketing research field, and Section 5 summarizes.

2. Thurstone scale in least squares and in regular maximum likelihood

Thurstone's law of comparative judgment describes psychological stimuli S_i $(i = 1, 2, \dots, m)$ as random normal variables $x_i = N(v_i, \sigma_i)$ with means v_i and standard deviations σ_i. Thurstone scaling presents the values v_i as the stimulus' position on the psychological continuum. The cumulative probability that one stimulus is preferred over another is:

$$p_{ij} = F(v_i - v_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-y^2/2}\,\mathrm{d}y, \tag{1}$$

where $z = (v_i - v_j)/\bar{\sigma}$ is a standardized variable with the standard deviation of the difference $\bar{\sigma} = (\sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j r_{ij})^{1/2}$, and with r_{ij} is the correlation between the ith and jth stimuli. Thurstone described several models, and the most commonly used is the Case-V model which corresponds to the assumption of equal standard deviations $\bar{\sigma}$ for all pairs of stimuli. This assumption is true if the variances of all stimuli are equal, and their correlations are also equal.

For the given values p_{ij} at the left-hand side (1) their quantiles, or z-values are:

$$z_{ij} = F^{-1}(p_{ij}) = v_i - v_j. \tag{2}$$

There are more equations (2) for the pairs $(i > j, j = 1, \dots, m-1)$ than m values v_i themselves. For estimation of the parameters v_i, Mosteller [33] suggested to minimize the Least Squares of the deviations ε_{ij} from the model $z_{ij} = v_i - v_j + \varepsilon_{ij}$:

$$\mathrm{LS}(v) = \sum_{i \neq j}^{m} (z_{ij} - (v_i - v_j))^2. \tag{3}$$

The objective (3) corresponds to a linear regression by dummy variables with the coefficients v_i and these coefficients need a shift identifiability constraint:

$$\sum_{i=1}^{m} v_i = 0. \tag{4}$$

Then the first order condition $\mathrm{dLS}/\mathrm{d}v_i = 0$ for minimizing (3) yields the estimate:

$$v_i = \frac{1}{m}\sum_{j=1}^{m} z_{ij} + \frac{1}{m}\sum_{j=1}^{m} v_j = \frac{1}{m}\sum_{j=1}^{m} z_{ij}. \tag{5}$$

Therefore, a position on the psychological scale for each ith item equals the mean value of the z-scores for the comparison of this item with the others.

In the practice of Thurstone scaling instead of the probabilities p_{ij} the sample proportions of each item's preference over the others are available. These frequencies are usually presented in a matrix where each element f_{ij} corresponds to preference of item j over item i. Then the frequencies are transformed into the quantiles of the normal distribution (2), so that $z_{ij} = F^{-1}(f_{ij})$. The averaged z-values (5) are used as the positions of the items on the Thurstone scale of preferences. They are usually reduced to the standard zero-one scale of preferences by the transformation:

$$\tilde{v}_j = \frac{v_j - \min(v)}{\max(v) - \min(v)}, \tag{6}$$

where min and max denote the minimum and maximum of all the v_j values. This approach is the standard Case V Thurstone scale estimation procedure.

Now we describe the maximum likelihood approach to individual paired comparison data, without using a frequency table. Consider a matrix X of paired comparisons, that has m columns of the variables $x_1, x_2, \ldots, x_m$ by the objects compared. The rows correspond to the different pair comparisons (total N of them) elicited from the respondents. In practice, usually a balanced plan is constructed so that each respondent compares several, but not all, possible pairs of the items. In any ith row, if the jth item is preferred to the kth one ($x_{ij} \succ x_{ik}$) then 1 and -1 are assigned to the jth and the kth columns, respectively:

$$\text{if } x_{ij} \succ x_{ik} \quad \text{then} \quad x_{ij} = +1 \text{ and } x_{ik} = -1. \tag{7}$$

All other entries in the row equal zero, and the matrix has a rank $m - 1$. Let us construct the vector $z = Xa$, where X is a design matrix (7), a is an mth order vector of the parameters of priority among the items, and z is the Nth order vector of scores for all pair comparisons. Each element of this vector equals the difference of the priorities between the compared items:

$$z_i = (X\,a)_i = x_{ij}a_j - x_{ik}a_k = \pm(a_j - a_k). \tag{8}$$

The Maximum Likelihood objective for the items of paired preference is:

$$\mathrm{ML} = \prod_{i=1}^{N} F(z_i), \tag{9}$$

where $F(z_i)$ is the probability that z_i is positive, or $x_{ij} \succ x_{ik}$. Maximization of the objective (9) for the normal and logistic distributions is considered in more detail in [24]. The solution can be reduced to the Iteratively Re-weighted Least Squares (IRLS) algorithm for solving nonlinear statistical problems [1,7,18,31].

In place of the direct ML calculations via an IRLS, the procedure (8) and (9) can be reduced to a binary response model regression model. Consider an extended matrix stacked by rows of the design matrix X (7) and the matrix $-X$ of the opposite formulation of the same statements. Construct a corresponding vector of binary output, with the first N elements equal one (vector e) and the other N elements equal zero (vector o). Then we get an extended vector of $2N$th order defined similarly to the scores (8) in the model:

$$\begin{pmatrix} e \\ o \end{pmatrix} = \begin{pmatrix} +X \\ -X \end{pmatrix} a, \tag{10}$$

where a is a vector of the priorities for Thurstone scaling. It is a binary response regression. The combined matrix in (10) has the rank $m - 1$, similarly to the matrix in (8), so in regression modelling we use just $m - 1$ columns of the matrix with one of the dummy variables excluded. The coefficient for this excluded variable will be obtained in place of the intercept in regression. After standardizing all of the coefficients to a 0–1 range (6) we obtain the Thurstone scale. Some other features of this approach are also discussed in [24].

Table 1
Example of a ranks matrix

Respondent i	Item 1	Item 2	Item 3	Item 4	Item 5
1	0	0	1	3	2
2	0	2	0	3	1
3	0	1	3	0	2
4	0	2	1	3	0
5	2	0	0	1	3
6	1	0	3	0	2
7	1	0	2	3	0
8	2	1	0	0	3
9	3	1	0	2	0
10	2	1	3	0	0

3. Order statistics in maximum likelihood for Thurstone scaling

Let us consider a more general approach permitting work with ranked or paired comparison data and expressing the Thurstonian scales via the covariates of influence. Suppose we have a task to evaluate the priorities of m items — for instance, brands, or product flavours in marketing research. In questioning N respondents we elicit the rank ordering of the items in the ascending order, from 1 for the worst one to m for the best item. Let now X be a matrix of N by m order with the entries x_{ij} of ranks that each ith respondent ($i = 1, 2, \ldots, N$) assigned to jth item ($j = 1, 2, \ldots, m$). In practice, the number m of the items can be too big (say, a couple of dozen) to ask the respondents to order all of them. Usually a balanced plan is constructed so that each respondent rank-orders just several $n (n \leq m)$ of the items. So the rank matrix can have in each row n filled positions (for the ranks from 1 for the worst item to n for the best one) and the others are zeros (actually, a different number of the entries, or all of them could be filled in each row as well). Let us also introduce a vector of parameters a that is an mth order vector of the items' priorities to be estimated.

The ranks data from each respondent we consider as a random sample and apply the results of the order statistics to it. To remind the needed relations of the order statistics, suppose $x_1, x_2, \ldots, x_n$ represent a random sample from a distribution with a continuous p.d.f. $f(x)$ and the corresponding cumulative probability $F(x)$. Let y_1 be the smallest of these x-s, then y_2 is the next one, etc., till y_n as the largest one, so $y_1 < y_2 < \cdots < y_n$. Each of these y_k (where $k = 1, 2, \ldots, n$) is called the kth order statistics of the random sample. The marginal p.d.f. of y_k in terms of $f(x)$ and $F(x)$ is defined by the expression:

$$g_k(y_k) = \frac{n!}{(k-1)!(n-k)!}[F(y_k)]^{k-1}[1 - F(y_k)]^{n-k} f(y_k). \tag{11}$$

To estimate the items' priorities a_j we suggest maximization of the likelihood criterion constructed by the marginal p.d.f. of the order statistics (11). Up to unimportant for maximization constant terms, the ML objective is:

$$\mathrm{ML} = \prod_{i=1}^{N}\prod_{k=1}^{n} g_k(y_{ik}) = \prod_{i=1}^{N}\prod_{k=1}^{n}[F(y_{ik})]^{k-1}[1 - F(y_{ik})]^{n-k} f(y_{ik}). \tag{12}$$

Each y_{ik} can be presented via the parameter a_j identified in each ith row by the jth column that contains the rank equaled k.

For an explicit illustration, let us take an artificial data of ranking three out of a total of five items. In a balanced design each item is shown to respondents six times. The ranks elicited from ten respondents are presented in the matrix shown in Table 1. The ranks 1, 2 and 3 correspond to the ordering of the items from the worst to the best one, and zero indicates the nonpresented items.

In the simplest case without covariates the order statistics y_{ik} (11) and (12) in each ith row can be substituted by the parameter a_k of the item preference, so $y_{ik} = a_k$. The rank k for the order statistic equals the value x_{ij} in each entry of the ranks matrix.

The first respondent in Table 1 ranked the items in the order $x_3 < x_5 < x_4$, so the related marginal p.d.f. (11) are:

$$\begin{cases} g_1(y_{11}) = \dfrac{3!}{0!\,2!}[F(a_3)]^0[1-F(a_3)]^2 f(a_3) \\ g_2(y_{12}) = \dfrac{3!}{1!\,1!}[F(a_5)]^1[1-F(a_5)]^1 f(a_5) \\ g_3(y_{13}) = \dfrac{3!3!}{2!\,0!}[F(a_4)]^2[1-F(a_4)]^0 f(a_4). \end{cases} \tag{13a}$$

In a similar pattern we construct all the other p.d.f., to the last one for the tenth respondent:

$$\begin{cases} g_1(y_{10,1}) = \dfrac{3!}{0!\,2!}[F(a_2)]^0[1-F(a_2)]^2 f(a_2) \\ g_2(y_{10,2}) = \dfrac{3!3!}{1!\,1!}[F(a_1)]^1[1-F(a_1)]^1 f(a_1) \\ g_3(y_{10,3}) = \dfrac{3!3!}{2!\,0!}[F(a_3)]^2[1-F(a_3)]^0 f(a_3). \end{cases} \tag{13b}$$

Total product of all p.d.f. presented in all the rows (13) constitutes the ML objective (12).

The logarithm of ML objective (12) can be presented in a general form as follows:

$$\ln \mathrm{ML} = \sum_{i=1}^{N}\sum_{j=1}^{m}\left\{(x_{ij}-1)[\ln F(a_j)] + (n-x_{ij})\ln[1-F(a_j)] + \ln f(a_j)\right\}\mathrm{sign}(x_{ij}), \tag{14}$$

where x_{ij} are the entries of a ranks matrix (as in Table 1), n is the maximum rank (that is less or equal the total number of items, $n \le m$). The summing in (14) includes all the items ($j = 1, \ldots, m$), but the sign function

$$\mathrm{sign}(x_{ij}) = \begin{cases} 1, & \text{if } x_{ij} > 0, \\ 0, & \text{if } x_{ij} = 0, \end{cases} \tag{15}$$

identifies the actual inputs from the nonzero entries of the ranks matrix. The maximization of the objective (14) can be performed by the IRLS procedure similar to that given in [24], or by the nonlinear optimizing algorithms available in the statistical packages of any modern software.

In the presence of covariates, when besides the matrix of ranks we have a matrix of the independent variables $u_{i1}, u_{i2}, \ldots, u_{iq}$ (for instance, demographical data of the respondents' age, education, income, etc.), each parameter a_j of the jth item preference on the Thurstonian scale can be extended by the aggregate:

$$a_j = b_{0j} + b_{1j}u_{i1} + b_{2j}u_{i2} + \cdots + b_{qj}u_{iq}. \tag{16}$$

In (16) a parameter a_j is defined via an intercept b_{0j} and other coefficients $b_{1j}, b_{2j}, \ldots, b_{qj}$ of summation, so the Thurstonian echelons can be presented by their dependence on the covariates. Using parameterization (16) we maximize the ordinal statistics objective (14) by total set of $b_{0j},\ b_{1j}, \ldots, b_{qj}$ values for all the ranked items ($j = 1, \ldots, m$). The procedure of optimization is similar to the IRLS technique described for the latent class regression modeling [25]. With parameterization (16) we can draw out Thurstone scale to series of graphs by each of the covariates, or to use other dimensional presentation by the covariates.

Thurstone modelling is usually considered in the normal distribution assumption (1). Besides the normal distribution with the density and cumulative probability as in (1), we can also use other links. For example, we can apply the logistic probability and density functions:

$$F(y_k) = \frac{\exp(y_k)}{1+\exp(y_k)}, \qquad f(y_k) = \frac{\mathrm{d}F(y_k)}{\mathrm{d}\,y_k} = \frac{\exp(y_k)}{[1+\exp(y_k)]^2} = F(y_k)[1-F(y_k)], \tag{17}$$

or the multinomial logit probability and density functions defined by the similar relations:

$$F(y_k) = \frac{\exp(y_k)}{1+\sum_{j=2}^{m}\exp(y_j)}, \qquad f(y_k) = \frac{\mathrm{d}F(y_k)}{\mathrm{d}\,y_k} = F(y_k)[1-F(y_k)], \tag{18}$$

Table 2
Thurstone preference: Estimation by several methods

Item	Thurstone scale regular (5)			Order statistics maximum likelihood							
				Normal (14)			Logit (17)			Multinom. (18)	
	Scale	Prob.	Share	Scale	Prob.	Share	Scale	Prob.	Share	Scale	Share
A	.708	.543	.136	.709	.538	.134	.709	.536	.134	.753	.135
B	.332	.428	.107	.328	.434	.109	.330	.437	.109	.376	.107
C	.934	.610	.152	.942	.600	.15	.942	.596	.149	.954	.152
D	0	.332	.083	0	.348	.087	0	.355	.089	0	.086
E	.165	.379	.095	.164	.390	.098	.165	.395	.099	.193	.096
F	.709	.543	.136	.710	.538	.135	.710	.536	.134	.754	.135
G	.700	.540	.135	.702	.536	.134	.702	.534	.134	.747	.134
H	1	.629	.157	1	.616	.154	1	.610	.153	1	.156

where $\exp(y_1) = 1$ is used to make the total of all probabilities equals one. It is interesting to note that with the logit functions (17) or (18) the order statistics p.d.f. (11) can be simplified and expressed via the cumulative probability only:

$$g_k(y_k) = \frac{n!}{(k-1)!(n-k)!}[F(y_k)]^k[1-F(y_k)]^{n-k+1}. \tag{19}$$

The order p.d.f. (11), or (19), can be interpreted in terms of the total p.d.f. split to the inputs proportional to the binomial probabilities of the order statistics. More explicitly, let us consider the expression (11) represented via the shifted parameters $\tilde{n} = n - 1$, $\tilde{k} = k - 1$:

$$g_k(y) = \left\{\frac{\tilde{n}\,!}{\tilde{k}!(\tilde{n}-\tilde{k})!}p^{\tilde{k}}(1-p)^{\tilde{n}-\tilde{k}}\right\}[(\tilde{n}+1)f(y)], \tag{20}$$

where we denote $p = F(y)$. The expression in figure parentheses (20) is the binomial p.d.f. with the probability p of success in $\tilde{k}$ out of $\tilde{n}$ trying. It means that each order statistics' input into the ML objective (12) is proportional to the probability of the success in having a higher rank k in the total set of all the items orderings by all respondents. We can also note that in the optimizing objective (12), or (14), a restriction of the (4) kind is not needed. Thus, the obtained preferences correspond to the cardinal scale of the probabilities to choose each of the items under consideration, not just to the ordinal scale of the probabilities' differences as obtained in the regular Thurstone scaling. The expression (20) also shows that the order statistics is actually an extension of a binary response model to the case of several ordinal echelons of the output represented by the ranked data.

Besides the ranks data used in the above described preference scaling, the pair comparison data can be also used in the order statistics approach. For a pair comparison data we apply the joint p.d.f. of each two order statistics, $y_k < y_j$ (see, for instance, [19]):

$$\begin{aligned} g_{kj}(y_k, y_j) = {} & \frac{n!}{(k-1)!(j-k-1)!(n-j)!} \\ & \times [F(y_k)]^{k-1}[F(y_j)-F(y_k)]^{j-k-1}[1-F(y_j)]^{n-j}f(y_k)f(y_j). \end{aligned} \tag{21}$$

Incorporating the p.d.f. (21) by all the paired comparisons and the respondent, we obtain a maximum likelihood objective similar to that in (12), or in (14), and optimize by its scaling parameters a_j of the items' preference, or by the parameters (16) extended by the covariates.

In practical problems of preference scaling, together with the data ordering and possible covariates there is often such data as appertaining to the importance of each observation, for instance, a purchase intention, or probability to take an action, etc. These data can be used as an additional weight in the order statistics' maximum likelihood formulation. The logarithm of ML objective (14) can be easily generalized to the weighted objective by incorporating the supplementary weights multiplied by the items of each ith observation.

Fig. 1. Thurstone scale in 0–1 interval.

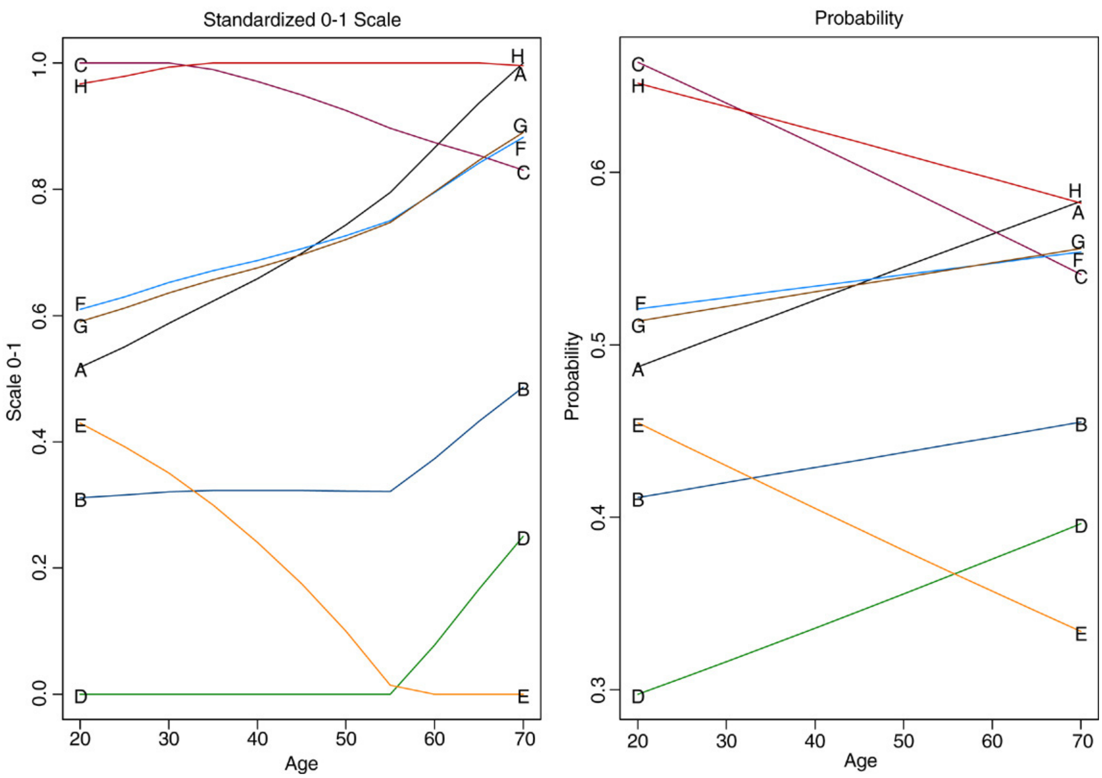

Fig. 2. Thurstone scaling: Trace by age.

4. Numerical modeling

For a numerical example we consider data from a real marketing research project where 513 respondents rank ordered eight flavours of a snack. The flavours are called: A — Au Gratin, B — Mashed Sweet Potatoes, C —

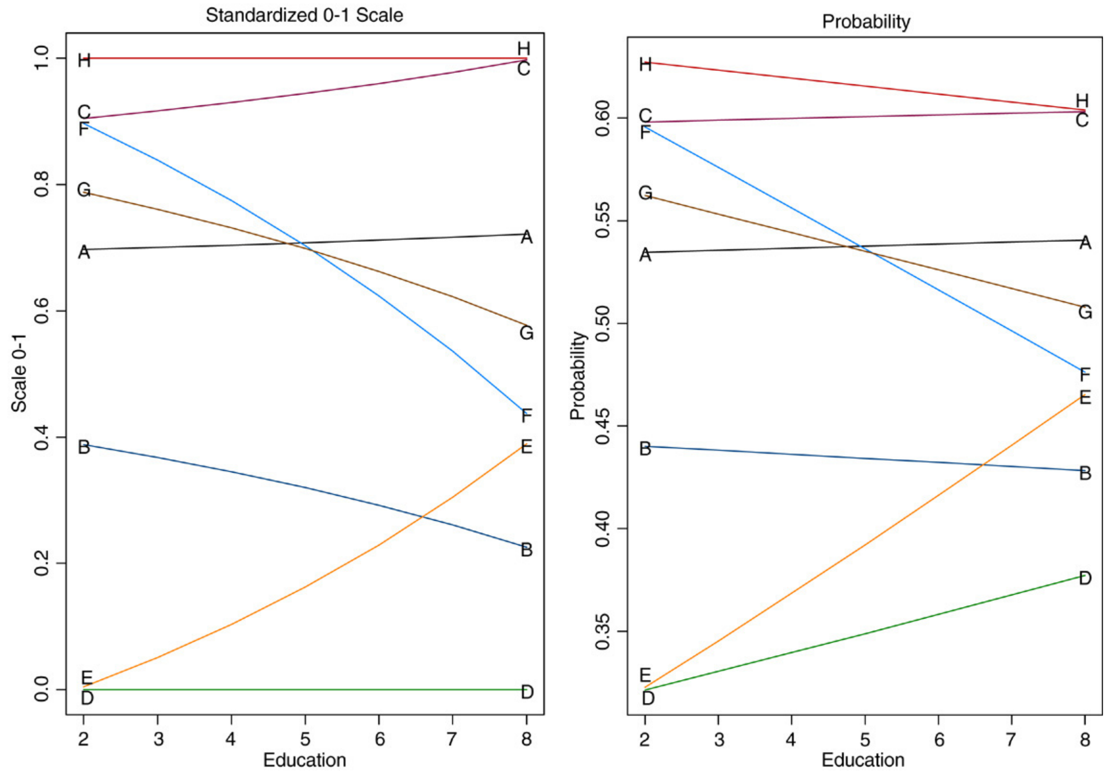

Fig. 3. Thurstone scaling: Trace by education.

Cheddar & Sour Cream, D — Yukon Gold, E — Rosemary & Garlic, F — Butter & Sour Cream, G — Sour Cream & Chives, H — Cheddar Cheese. Besides the ranks by these flavours, there were several demographic variables. For instance, age was measured for the respondents from 18 to 65 and older. Education was scaled as: 1 = Grade school or less, 2 = Some high school, 3 = High school graduate, 4 = Some college, 5 = Two-year college/technical school, 6 = Four-year college, 7 = Some postgraduate work, and 8 = Postgraduate degree. And Income in $US was echeloned from less than 25K to more than 120K per household per year.

Table 2 contains the results of Thurstone estimations in different approaches to the flavours' positioning on the psychological continuum. The first three numeric columns in Table 2 present the values of the regular Thurstone scaling evaluated by the paired preferences derived from the ranked data. There are the scale normalized to 0–1 interval (6), the probability (1) corresponding to the found quantiles, and the shares of these probabilities. The next three columns present the results of the order statistics ML (14) obtained with the normal distribution — there are also the scale in the standard 0–1 interval, the probability of each item choice, and the shares of the probabilities. Then the analogue results of the order statistics ML obtained with the logistic (17) distribution are shown. The last two columns contain the results of the multinomial (18) link used in the order statistics ML, when we obtain the scale normalized by (6), and the shares corresponding to the probability of each item's choice. The scales by all the approaches are very similar to the regular Thurstone scaling, that can be presented in Fig. 1. The probabilities and their shares are also similar by different methods, that confirms the possibility to apply the order statistics maximum likelihood technique directly to the ranks data.

A thorough Thurstone scaling by a covariate (16) can reveal a more complicated behaviour of the preferences dependence on another variable. Applying the order statistics ML with the normal distribution (14) to the ranks data with an added covariate, we obtain the coefficients of the aggregate (16), and use them to produce the scale levels and probabilities of the items' choice dependent on a covariate. These results are presented in Figs. 2–4 — each of them shows the standardized 0–1 Thurstonian scale and the probability of each flavour choice traced by a covariate.

In Fig. 1 we saw that the best choice across all the respondents is the flavour H, then goes C, etc., to the worst flavor D. In Fig. 2 we see now that the flavour H becomes the leading choice only for the respondents from about

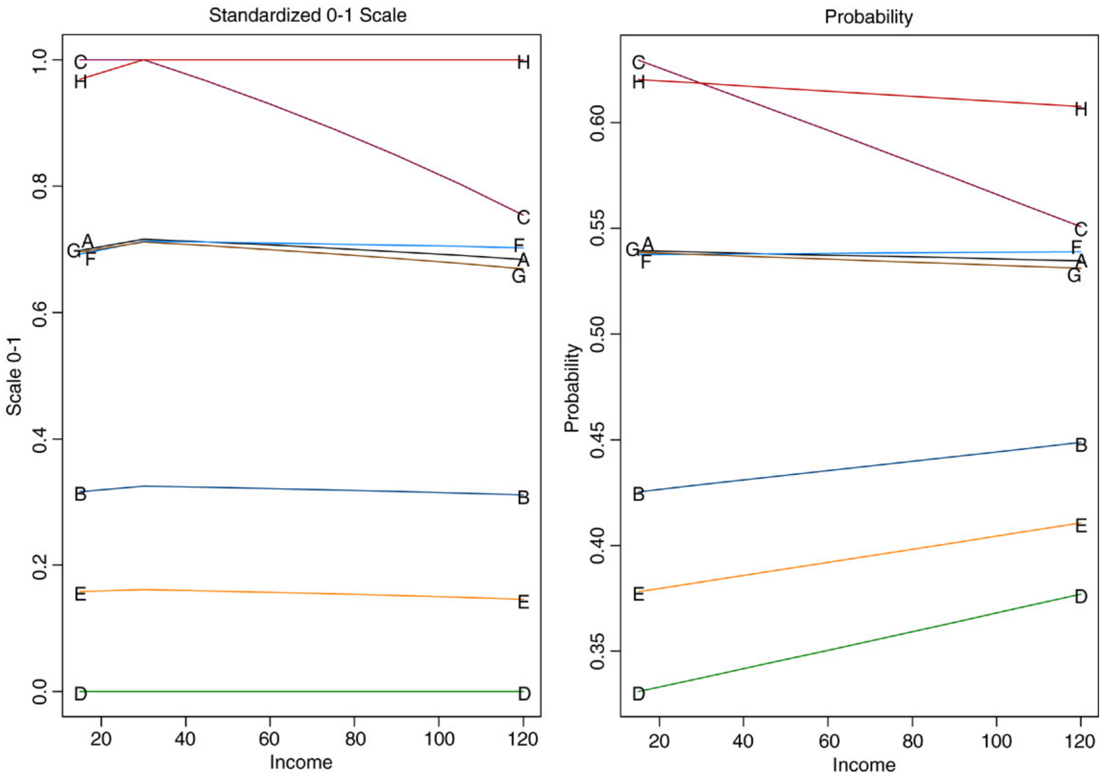

Fig. 4. Thurstone scaling: Trace by income.

30 years old, while for the younger group the best choice is the flavour C. Although the latter one begins to be less attractive than even A, G, and F flavours for the elder consumers of 55–60 and more years old. Also, the appeal of the flavour E falls with the age increase and it becomes the worst flavour for the customers from 55 years old. The Fig. 2 right-hand side window of the probability graphs produces more smooth behaviour of the flavours' choice by the age, and permits to study the consumers behaviour in detail.

The next graphs in Fig. 3 demonstrate the specifics of the preference among the flavours along the scale of the respondents education. We can indicate the change of the priorities that occurs at the level 5 of two-year college/technical school and of more educated persons. Fig. 4 presents the behaviour of the consumers on the basis of their income (in $K), and shows a change in the pattern at around $30K. It is interesting to note that by all the covariates the probability of all the choices become closer, so for the elder, more educated persons with a higher income the differences among the product flavors become less distinctive.

5. Summary

We considered Thurstonian preference scaling via the order statistics maximum likelihood evaluation by the individual ranking data with additional covariates. We presented this approach as an Iteratively Reweighted Least Squares procedure applied to normal, logistic, or multinomial probability distribution. The described methods enrich possibilities of rank or paired comparison modelling, and are convenient and useful for theoretical research and practical applications of Thurstone scaling. Future development of the suggested approach we see in the hierarchical Bayesian estimations with mix effects of the covariates, that can be readily achieved in the developed approach of the order statistics maximum likelihood optimization.

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