Homework 5

Problem 1 Consider the process $\{\varepsilon(\boldsymbol{s}) : \boldsymbol{s} \in D\}$ such that $E(\varepsilon(\boldsymbol{s})) = 0$, $\text{var}(\varepsilon(\boldsymbol{s})) = \sigma^2$, and if $\boldsymbol{s}_1 \neq \boldsymbol{s}_2$ then $\varepsilon(\boldsymbol{s}_1)$ is uncorrelated with $\varepsilon(\boldsymbol{s}_2)$. Find the variogram $2\gamma(\boldsymbol{h})$ of the process $\varepsilon(\cdot)$.

Solution. For $\boldsymbol{s}_1 \neq \boldsymbol{s}_2$ we have

$$\begin{aligned}
2\gamma(\boldsymbol{s}_1 - \boldsymbol{s}_2) &= \text{var}(Z(\boldsymbol{s}_1) - Z(\boldsymbol{s}_2)) \\
&= \text{var}(Z(\boldsymbol{s}_1)) + \text{var}(Z(\boldsymbol{s}_2)) - 2\text{cov}(Z(\boldsymbol{s}_1), Z(\boldsymbol{s}_2)) \\
&= \sigma^2 + \sigma^2 - 0 \\
&= 2\sigma^2.
\end{aligned}$$

Hence,

$$2\gamma(\boldsymbol{h}) = \begin{cases} 2\sigma^2 & \text{if } \boldsymbol{h} \neq \boldsymbol{0}, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2 Consider the process $\{\varepsilon(s) : s \geq 0\}$, defined in one dimension, as in question 1. Let

$$W(s) = \int_0^s \varepsilon(u)du\,; \qquad s \geq 0\,.$$

(a) Show that $E(W(s)) = 0$ and $\text{cov}(W(s), W(t)) = \sigma^2 \min(s, t)$.

Solution.
We have

$$\begin{aligned}
E[W(s)] &= E\left[\int_0^s \varepsilon(u)du\right] = \int_0^s E[\varepsilon(u)]du \\
&= \int_0^s 0du = 0.
\end{aligned}$$

Assume $s > t$, then

$$\begin{aligned}
\text{cov}(W(s), W(t)) &= \text{cov}(W(t) + (W(s) - W(t)), W(t)) \\
&= \text{cov}(W(t), W(t)) + \text{cov}(W(s) - W(t), W(t)) \\
&= \text{var}(W(t)),
\end{aligned}$$

since $W(s) - W(t) = \int_t^s \varepsilon(u)du$ is independent of $W(t) = \int_0^t \varepsilon(u)du$. Now,

$$\begin{aligned}\text{var}(W(t)) &= \text{var}\left(\int_0^t \varepsilon(u)du\right) = \int_0^t \text{var}(\varepsilon(u))du = \int_0^t \sigma^2 du \\ &= \sigma^2 t.\end{aligned}$$

Hence, in general, $\text{cov}(W(s), W(t)) = \min(s,t)\sigma^2$.

(b) Find $\text{var}(W(s) - W(t))$.

Solution.
Assume $s > t$, then

$$\begin{aligned}\text{var}(W(s) - W(t)) &= \text{var}\left(\int_t^s \varepsilon(u)du\right) = \int_t^s \text{var}(\varepsilon(u))du \\ &= \sigma^2(s-t),\end{aligned}$$

similarly, if $s < t$, then $\text{var}(W(s) - W(t)) = \sigma^2(t-s)$. Hence, in general, $\text{var}(W(s) - W(t)) = \sigma^2|s-t|$.

(c) Comment on whether the process $W(\cdot)$ is second-order stationary or intrinsically stationary.

Solution.
Since $\text{var}(W(s) - W(t)) = \sigma^2|s-t|$ and depends therefore only on the distance between s and t, $W(\cdot)$ is intrinsically stationary. On the other hand, $\text{cov}(W(s), W(t)) = \min(s,t)\sigma^2$ (i.e., does not *only* depend on the distance between $s - t$), so the process is not second-order stationary.

Problem 3 Consider a spatial random process $\{Z(\boldsymbol{s}) : \boldsymbol{s} \in D\}$ and assume that its mean and covariance function are *known*. That is, assume that μ and $C(\cdot,\cdot)$ are

$$\begin{aligned}\mu &\equiv E(Z(\boldsymbol{s}))\,; & \boldsymbol{s} &\in D \\ C(\boldsymbol{s},\boldsymbol{u}) &\equiv \text{cov}(Z(\boldsymbol{s}), Z(\boldsymbol{u}))\,; & \boldsymbol{s},\boldsymbol{u} &\in D\,.\end{aligned}$$

Suppose we want to predict $Z(\boldsymbol{s}_0)$ from data $\boldsymbol{Z} \equiv (Z(\boldsymbol{s}_1), \ldots, Z(\boldsymbol{s}_n))'$ using the linear predictor

$$Z^*(\boldsymbol{s}_0) = \sum_{i=1}^n \ell_i Z(\boldsymbol{s}_i) + c\,.$$

(a) Find $\ell_1, \ldots, \ell_n$, c to minimize

$$\text{MSPE} \equiv E(Z(\boldsymbol{s}_0) - Z^*(\boldsymbol{s}_0))^2 .$$

Solution. Let $\boldsymbol{\ell} \equiv (\ell_1, \ldots, \ell_n)'$, then $Z(\boldsymbol{s}_0) = \boldsymbol{\ell}'\boldsymbol{Z} + c$ and

$$\begin{aligned}
\text{MSPE} &= E[Z(\boldsymbol{s}_0) - (\boldsymbol{\ell}'\boldsymbol{Z} + c)]^2 \\
&= \text{var}(Z(\boldsymbol{s}_0) - (\boldsymbol{\ell}'\boldsymbol{Z} + c)) + (E[Z(\boldsymbol{s}_0)] - E[\boldsymbol{\ell}'\boldsymbol{Z} + c])^2 \\
&= \text{var}(Z(\boldsymbol{s}_0) - (\boldsymbol{\ell}'\boldsymbol{Z} + c)) + (\mu - (\boldsymbol{\ell}'\boldsymbol{\mu} + c))^2.
\end{aligned}$$

Now,

$$\begin{aligned}
\text{var}(Z(\boldsymbol{s}_0) - (\boldsymbol{\ell}'\boldsymbol{Z} + c)) &= \text{var}(Z(\boldsymbol{s}_0)) + \text{var}(\boldsymbol{\ell}'\boldsymbol{Z} + c) - 2\text{cov}(Z(\boldsymbol{s}_0), \boldsymbol{\ell}'\boldsymbol{Z} + c) \\
&= C(\boldsymbol{s}_0, \boldsymbol{s}_0) + \boldsymbol{\ell}'\Sigma\boldsymbol{\ell} - 2\boldsymbol{\ell}'\boldsymbol{c},
\end{aligned}$$

where $\boldsymbol{c} \equiv (C(\boldsymbol{s}_0, \boldsymbol{s}_1), \cdots, C(\boldsymbol{s}_0, \boldsymbol{s}_n))$ and Σ is the covariance matrix ($\Sigma_{ij} = C(\boldsymbol{s}_i, \boldsymbol{s}_j)$). Hence,

$$\text{MSPE} = C(\boldsymbol{s}_0, \boldsymbol{s}_0) + \boldsymbol{\ell}'\Sigma\boldsymbol{\ell} - 2\boldsymbol{\ell}'\boldsymbol{c} + (\mu - (\boldsymbol{\ell}'\boldsymbol{\mu} + c))^2$$

Taking the partial derivative gives,

$$\begin{aligned}
\frac{\partial \text{MSPE}}{\partial \boldsymbol{\ell}} &= -2\boldsymbol{c} + 2\Sigma\boldsymbol{\ell} - 2(\mu - \boldsymbol{\ell}'\mathbf{1}\mu - c) \\
\frac{\partial \text{MSPE}}{\partial c} &= -2(\mu - \boldsymbol{\ell}'\mathbf{1}\mu - c).
\end{aligned}$$

Setting both equal to zero and plugging the second equation into the first one gives $-2\boldsymbol{c} + 2\Sigma\boldsymbol{\ell} = 0$, or $\boldsymbol{\ell} = \Sigma^{-1}\boldsymbol{c}$, and then $c = \mu(\mathbf{1} - \boldsymbol{\ell}'\mathbf{1})$.
Using this,

$$\begin{aligned}
Z^*(\boldsymbol{s}_0) &= \boldsymbol{c}'\Sigma^{-1}(\boldsymbol{Z} - \mu\mathbf{1}) + \mu, \\
\text{MSPE} &= C(\boldsymbol{s}_0, \boldsymbol{s}_0) - \boldsymbol{c}'\Sigma^{-1}\boldsymbol{c}.
\end{aligned}$$

Finally, check if $\boldsymbol{\ell} = \Sigma^{-1}\boldsymbol{c}$ and $c = \mu(\mathbf{1} - \boldsymbol{\ell}'\mathbf{1})$ minimize MSPE.

(b) Notice that there are no restrictions in (i) to guarantee that $Z^*(\boldsymbol{s}_0)$ is unbiased. Show that the solution to (i) is indeed unbiased; that is, show

$$E(Z^*(\boldsymbol{s}_0)) = E(Z(\boldsymbol{s}_0)) .$$

Solution. We have,

$$\begin{aligned}
E[Z^*(\boldsymbol{s}_0)] &= E[\boldsymbol{c}'\Sigma^{-1}(\boldsymbol{Z} - \mu\mathbf{1}) + \mu] = \boldsymbol{c}'\Sigma^{-1}(E[\boldsymbol{Z}] - \mu\mathbf{1}) + \mu \\
&= \boldsymbol{c}'\Sigma^{-1}(\mu\mathbf{1} - \mu\mathbf{1}) + \mu = \mu \\
&= E[Z(\boldsymbol{s}_0)].
\end{aligned}$$