Tweedie New Researcher Invited Lecture

Poisson Disorder Problems

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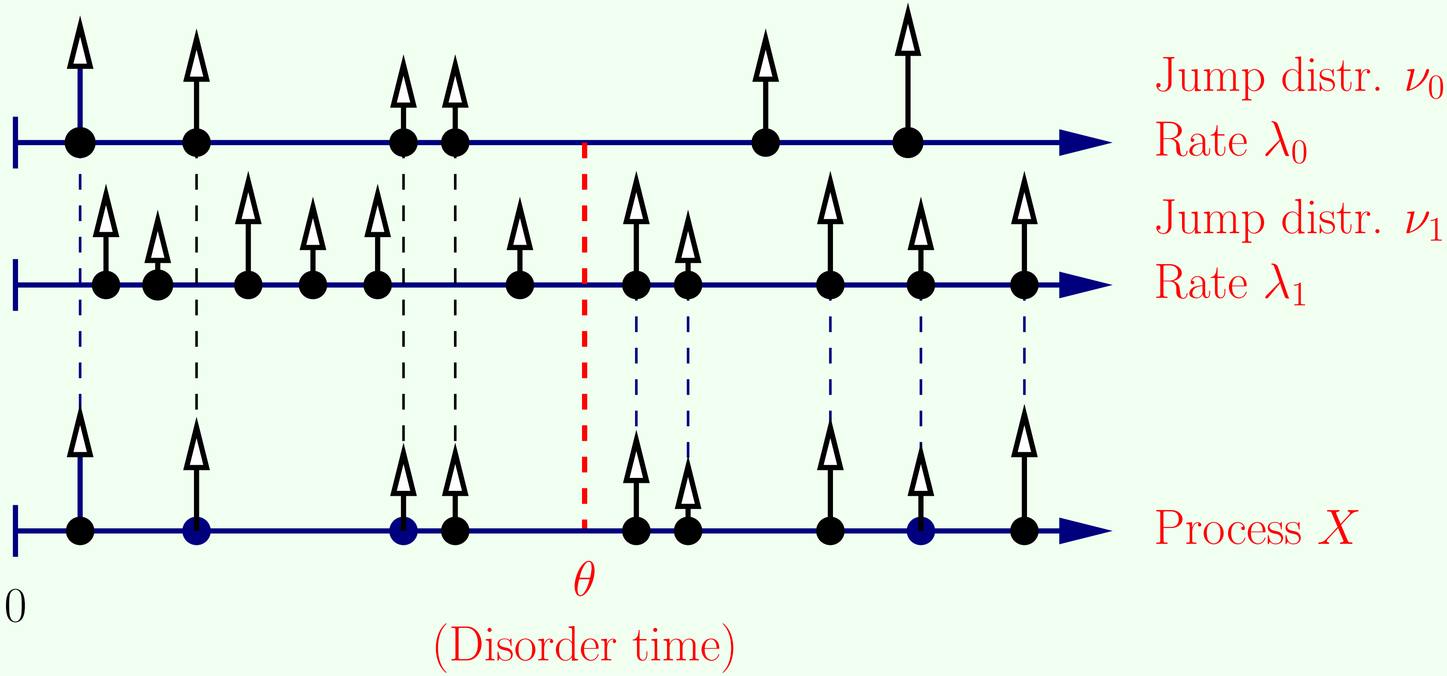
1. Problem Description

Let X be a **compound Poisson process** whose rate λ_0 and jump distribution $\nu_0(\cdot)$ change to λ_1 and $\nu_1(\cdot)$, respectively, at some **unknown** and **unobservable** time θ.



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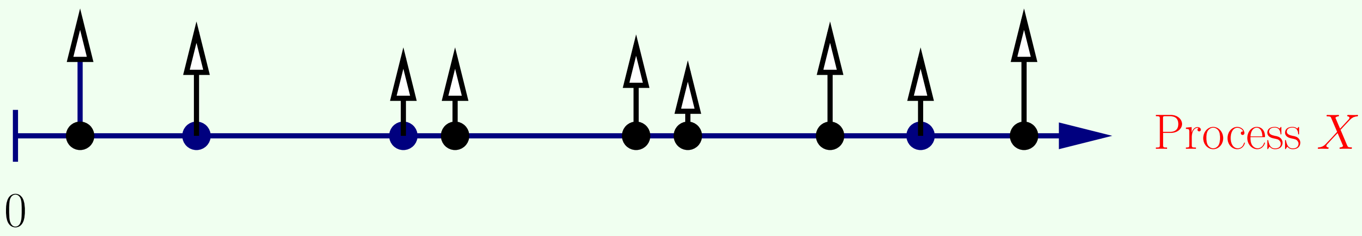
(Only the process X is observable.)



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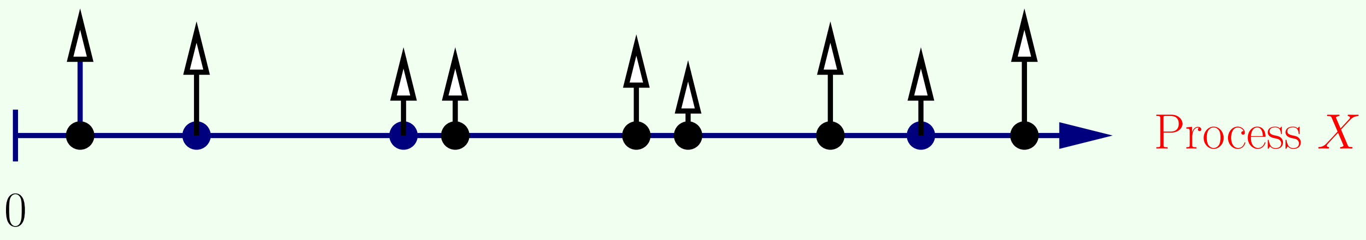

Problem: Find a **decision rule** which

- detects the disorder time θ as quickly as possible,
- is adapted to the history of X.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting random variables θ, $Y_1, Y_2, \cdots$, a counting process $N = \{N_t; t \geq 0\}$. Define

$$X_t = X_0 + \sum_{k=1}^{N_t} Y_k \equiv X_0 + \int_{(0,t]\times\mathbb{R}^d} y\, p(ds, dy), \quad t \geq 0$$

in terms of the point process describing jump times and sizes

$$p((0,t] \times A) \triangleq \sum_{k=1}^{\infty} 1_{\{\sigma_k \leq t\}} 1_{\{Y_k \in A\}}, \quad t \geq 0,\ A \in \mathcal{B}(\mathbb{R}^d).$$

and $\sigma_k = \inf\{t > \sigma_{k-1} : X_t \neq X_{t-}\}$, $k = 1, 2, \ldots$ $(\sigma_0 \equiv 0)$.

(1)
$$\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0} \quad \text{as the natural filtration of } X,$$
$$\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}, \quad \mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma\{\theta\}.$$

The disorder time θ has the distribution

(2) $$\mathbb{P}\{\theta = 0\} = \pi \quad \text{and} \quad \mathbb{P}\{\theta > t | \theta > 0\} = e^{-\lambda t}, \quad t \geq 0.$$

The counting process $\{p(t, A) \triangleq p((0,t] \times A); t \geq 0\}$ is a non-homogeneous Poisson process with the $(\mathbb{P}, \mathbb{G})$-intensity

(3) $$h(t, A) \triangleq \lambda_0 \nu_0(A) 1_{\{t<\theta\}} + \lambda_1 \nu_1(A) 1_{\{t\geq\theta\}}, \quad t \geq 0.$$

Our **problem** is (i) to calculate the **minimum Bayes risk**

$$(4)\quad \begin{aligned} V(\pi) &\triangleq \inf_{\tau\in\mathbb{F}} R_\tau(\pi), \\ R_\tau(\pi) &\triangleq \mathbb{P}\{\tau<\theta\} + c\cdot\mathbb{E}\left[(\tau-\theta)^+\right], \quad \pi\in[0,1), \end{aligned}$$

and (ii) to find an $\mathbb{F}$-stopping time τ where the infimum is attained (if exists, called a **minimum Bayes detection rule**).

The **Bayes risk** $R_\tau(\pi)$ in (4) associated with every $\mathbb{F}$-stopping time τ is the sum of

- the false alarm frequency $\mathbb{P}\{\tau<\theta\}$, and
- the expected detection delay cost $c\cdot\mathbb{E}[(\tau-\theta)^+]$.

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Standard Bayes risks include

Linear delay penalty: $R_\tau(\pi) = \mathbb{P}\{\tau<\theta\} + c\,\mathbb{E}[(\tau-\theta)^+]$,

$R_\tau^{(\varepsilon)}(\pi) \triangleq \mathbb{P}\{\tau<\theta-\varepsilon\} + c\,\mathbb{E}[(\tau-\theta)^+]$,

Expected miss: $R_\tau^{(\text{miss})}(\pi) \triangleq \mathbb{E}[(\theta-\tau)^+] + c\,\mathbb{E}[(\tau-\theta)^+]$,

Expon. delay penalty: $R_\tau^{(\text{exp})}(\pi) \triangleq \mathbb{P}\{\tau<\theta\} + c\,\mathbb{E}[e^{\alpha(\tau-\theta)^+}-1]$.

Where do the disorder problems arise?

Insurance companies: Recalculate the premiums for the future sales of insurance policies when the risk structure changes (e.g., the arrival rate of claims of certain size).

Airlines, retailers of perishable products: Adjust the prices when a change in the demand structure is detected (e.g., the arrival rate of a certain type of customers).

Quality control and maintenance: Inspect, recalibrate, or repair tools and machines as soon as a manufacturing process goes out of control.

Fraud and computer intrusion detection: Alert the inspectors for an immediate investigation as soon as abnormal credit card activity, cell phone calls, or computer network traffic are detected.

2. The Model

Let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a p.s. with independent random elements:

- a Poisson process $N = \{N_t; t \geq 0\}$ with rate λ_0,
- iid $\mathbb{R}^d$-valued rv's $Y_1, Y_2, \ldots$ with distr. $\nu_0(\cdot)$ $(\nu_0(\{0\}) = 0)$,
- a rv θ with the distribution

$$\mathbb{P}_0\{\theta = 0\} = \pi \quad \text{and} \quad \mathbb{P}_0\{\theta > 0\} = (1-\pi)e^{-\lambda t},\ t \geq 0.$$

A compound Poisson process with arrival rate λ_0 and jump distribution $\nu_0(\cdot)$ is defined by

$$X_t = X_0 + \sum_{k=1}^{N_t} Y_k = X_0 + \int_{(0,t]\times A} y\, p(ds, dy), \quad t \geq 0$$

in terms of the point process on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^d))$

$$p((0,t]\times A) \triangleq \sum_{k=1}^{\infty} 1_{\{\sigma_k \leq t\}} 1_A(Y_k), \qquad t \geq 0,\ A \in \mathcal{B}(\mathbb{R}^d).$$

Under $\mathbb{P}_0$ the process $\{p((0,t]\times A); t \geq 0\}$ is homogeneous Poisson process with the $\mathbb{F}$-intensity $\lambda_0 \cdot \nu_0(A)$. Each σ_k is a jump time of X, and $\mathbb{F}$ is its history, and $\mathbb{G} = \mathbb{F} \vee \sigma\{\theta\}$.

Let λ_1 be a constant, and $\nu_1(\cdot)$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ absolutely continuous wrt $\nu_0(\cdot)$ with RN-derivative

$$f(y) \triangleq \frac{d\nu_1}{d\nu_0}(y), \qquad y \in \mathbb{R}^d.$$

Define locally a new probability measure $\mathbb{P}$ on $(\Omega, \vee_{t\geq 0}\mathcal{G}_t)$ by the Radon-Nikodym derivatives

$$(5) \quad \left.\frac{d\mathbb{P}}{d\mathbb{P}_0}\right|_{\mathcal{G}_t} = 1_{\{t<\theta\}} + 1_{\{t\geq\theta\}} e^{-(\lambda_1-\lambda_0)(t-\theta)} \prod_{k=N_{\theta-}+1}^{N_t} \left[\frac{\lambda_1}{\lambda_0} f(Y_k)\right], \; t \geq 0.$$

Then every counting process $\{p((0,t]\times A);\; t\geq 0\}$, $A \in \mathcal{B}(\mathbb{R}^d)$ is a nonhomogeneous Poisson process with the $(\mathbb{P}, \mathbb{G})$-intensity

$$(3) \qquad h(t, A) = \lambda_0 \nu_0(A) 1_{\{t<\theta\}} + \lambda_1 \nu_1(A) 1_{\{t\geq\theta\}}.$$

Since $\mathbb{P}_0 \equiv \mathbb{P}$ on $\mathcal{G}_0 = \sigma\{\theta\}$, the disorder time θ has the same distribution under $\mathbb{P}_0$ and $\mathbb{P}$.

Therefore, the model under the measure $\mathbb{P}$ of (5) has the same setup described in the beginning.

3. A Markovian sufficient statistic for detection problem

The Bayes risk $R_\tau(\pi) = \mathbb{P}\{\tau < \theta\} + \mathbb{E}\left[(\tau-\theta)^+\right]$, $\pi \in [0,1)$ in (4) for every $\mathbb{F}$-stopping rule τ can be written as

(6) $$R_\tau(\pi) = 1 - \pi + c\,(1-\pi)\,\mathbb{E}_0\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t - \frac{\lambda}{c}\right)dt\right].$$

The expectation in (6) is taken under the ref. p.m. $\mathbb{P}_0$, and

(7) $$\Phi_t \triangleq \frac{\mathbb{P}\{\theta \le t | \mathcal{F}_t\}}{\mathbb{P}\{\theta > t | \mathcal{F}_t\}}, \qquad t \in \mathbb{R}_+.$$

The process Φ is a piecewise-deterministic Markov process:

$$\left\{\begin{array}{ll} \Phi_t = x\left(t-\sigma_{n-1}, \Phi_{\sigma_{n-1}}\right), & t \in [\sigma_{n-1}, \sigma_n) \\ \Phi_{\sigma_n} = \dfrac{\lambda_1}{\lambda_0} f(Y_n)\Phi_{\sigma_n-} & \end{array}\right\}, \qquad n \ge 1.$$

The fucntion $x(\cdot,\phi) = \{x(t,\phi);\ t \ge 0\}$ is the solution of

$$\frac{d}{dt}x(t,\phi) = \lambda + a x(t,\phi), \quad t \in \mathbb{R}, \qquad \text{and} \quad x(0,\phi) = \phi; \text{ i.e.,}$$

$$x(t,\phi) = \phi_d + e^{at}\left[\phi - \phi_d\right], \quad t \in \mathbb{R}.$$

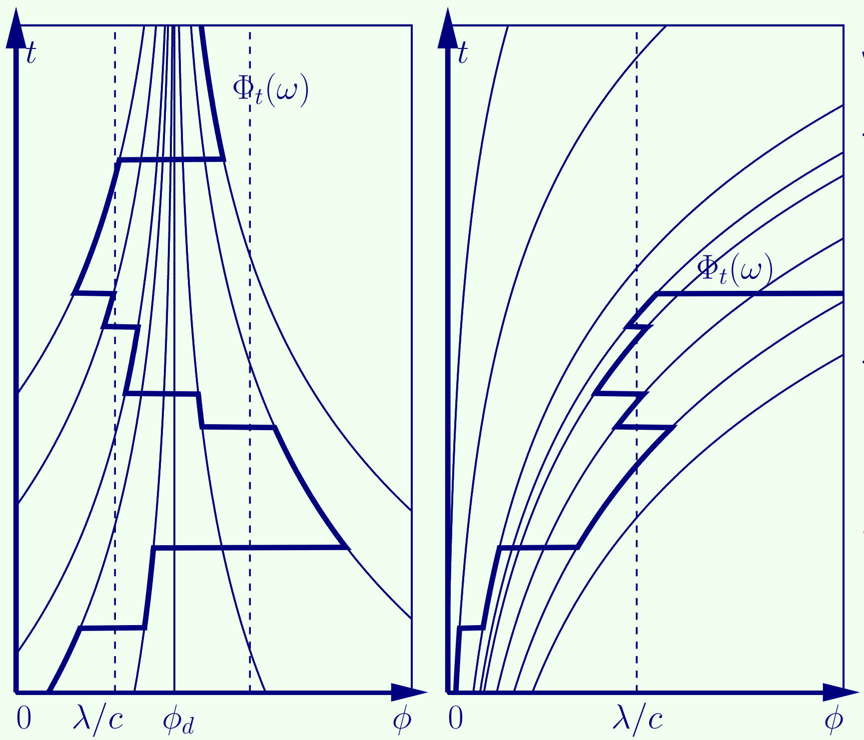
Here $a \triangleq \lambda - \lambda_1 + \lambda_0$, $\phi_d \triangleq -\lambda/a$.

The min. Bayes risk in (4) of the Poisson disorder problem is

$$U(\pi) = 1 - \pi + c\,(1-\pi)\cdot V\left(\frac{\pi}{1-\pi}\right), \qquad \pi \in [0,1).$$

The function $V : \mathbb{R}_+ \mapsto (-\infty, 0]$ is the value function of the discounted optimal stopping problem

$$(8) \qquad V(\phi) \triangleq \inf_{\tau\in\mathbb{F}} \mathbb{E}_0\left[\int_0^\tau e^{-\lambda t} g(\Phi_t)\,dt \,\middle|\, \Phi_0 = \phi\right]$$



(a) $\phi_d > 0$ (b) $\phi_d < 0$

with the running cost function

$$g(\phi) \triangleq \phi - \frac{\lambda}{c}, \qquad \phi \geq 0.$$

for the piecewise deterministic Markov process Φ.

[*Left: sample paths of the process* Φ]

4. Successive approximations

Let us introduce the family of optimal stopping problems

$$(9) \qquad V_n(\phi) \triangleq \inf_{\tau \in \mathbb{F}} \mathbb{E}_0^{\phi} \left[\int_0^{\tau \wedge \sigma_n} e^{-\lambda s} g(\Phi_s) ds \right], \qquad \phi \in \mathbb{R}_+,\ n \geq 0,$$

obtained from (8) by stopping the process Φ at the nth jump time σ_n of the process X.

Proposition. For every $n \geq 0$ and $\phi \in \mathbb{R}_+$, we have

$$(10) \qquad -\frac{1}{c} \cdot \left(\frac{\lambda_0}{\lambda + \lambda_0} \right)^n \leq V(\phi) - V_n(\phi) \leq 0.$$

Proof. Due to the discounting and exponentially distributed jump interarrival times of X under $\mathbb{P}_0$. □

Lemma. For every $\mathbb{F}$-stopping time τ and $n \geq 0$, there is an $\mathcal{F}_{\sigma_n}$-measurable random variable $R_n : \Omega \mapsto [0, \infty]$ such that

$$\tau \wedge \sigma_{n+1} = (\sigma_n + R_n) \wedge \sigma_{n+1}, \quad \mathbb{P}_0\text{-a.s. on} \quad \{\tau \geq \sigma_n\}.$$

If for every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}$ we define

$$Jw(t,\phi) = \int_0^t e^{-(\lambda+\lambda_0)u} \big(g + \lambda_0 \cdot Sw\big)\big(x(u,\phi)\big) du, \qquad t \in [0,\infty]$$

where $\quad Sw(x) \triangleq \int_{\mathbb{R}^d} w\left(\frac{\lambda_1}{\lambda_0} f(y)\, x\right) \nu_0(dy), \qquad x \in \mathbb{R}.$

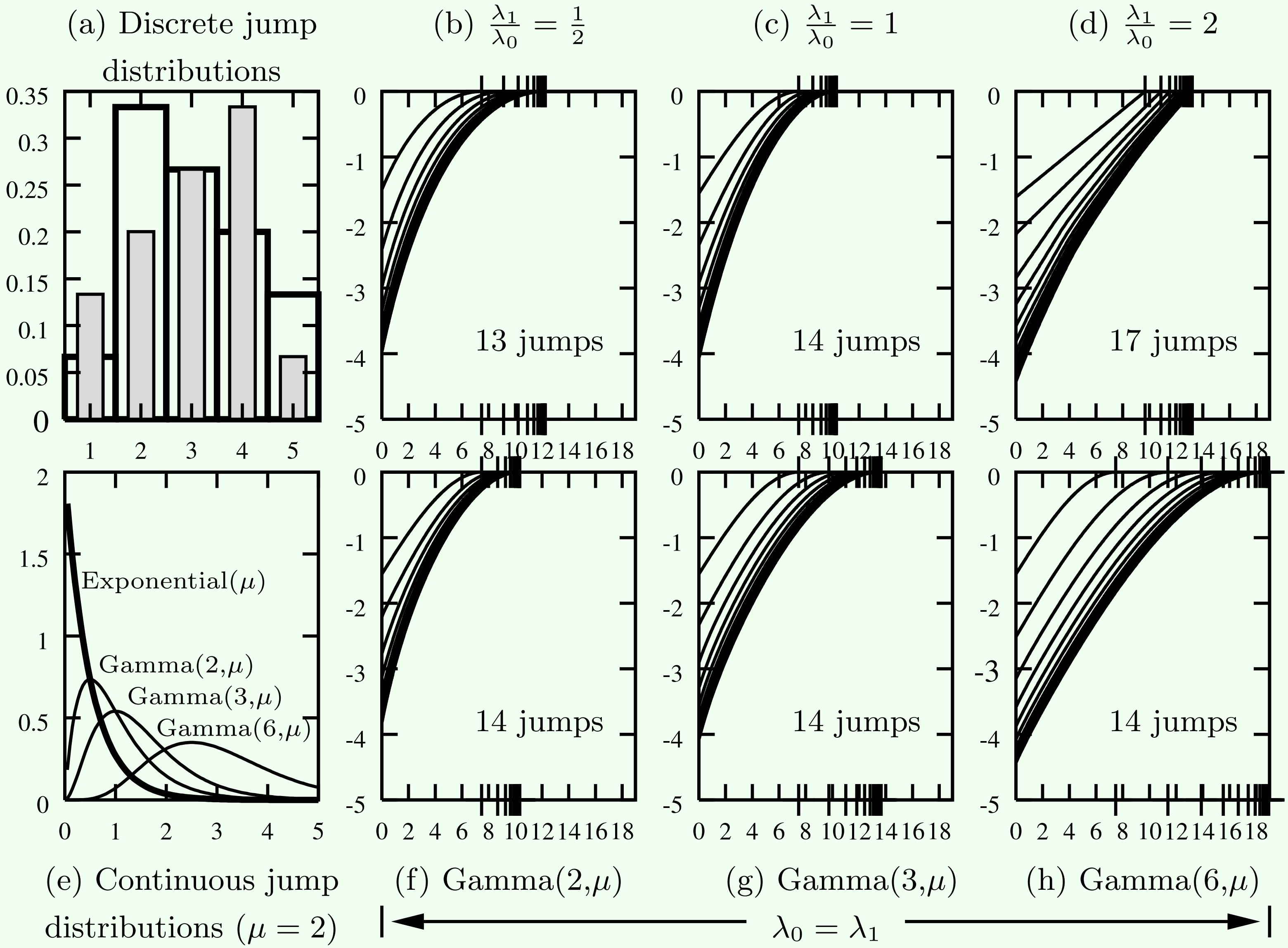
then we can calculate the successive approximations $\{V_n(\cdot)\}_{n\geq 1}$ of the value function $V(\cdot)$ by

$$V_0(\cdot) \equiv 0, \quad \text{and} \quad V_n(\cdot) = J_0 V_{n-1}(\cdot) \triangleq \inf_{t\geq 0} JV_{n-1}(t,\cdot) \quad \forall\, n \geq 1.$$

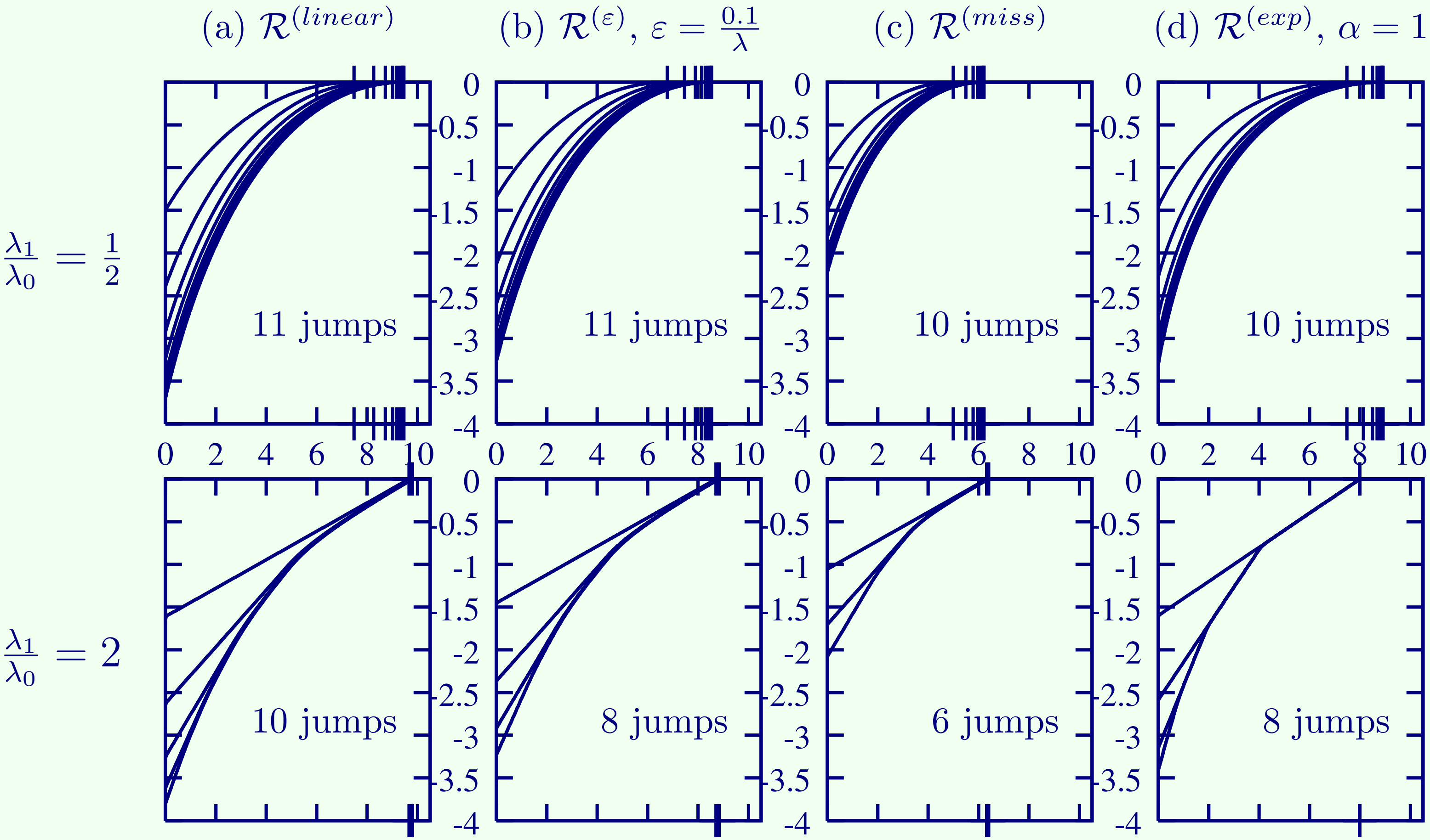
Moreover

1. $V_n(\cdot) \searrow V(\cdot)$ (exponentially fast)
2. $V(\cdot) = J_0 V(\cdot)$ on $\mathbb{R}_+$. (Dynamic programming equation)
3. The value function $V(\cdot)$ is concave and nonpositive.
4. The stopping region $\mathbf{\Gamma} = \{\phi \in \mathbb{R}_+ : V(\phi) = 0\}$ is in the form $\mathbf{\Gamma} = [\xi, \infty)$ for some $0 < \xi < +\infty$.

5. Examples



Parameters: $c = 0.2$, $\lambda = 1.5$, $\lambda_0 = 3$.



$$
\left.
\begin{aligned}
R_\tau^{(\text{linear})}(\pi) &\triangleq \mathbb{P}\{\tau < \theta\} + c\,\mathbb{E}(\tau-\theta)^+,\\
R_\tau^{(\varepsilon)}(\pi) &\triangleq \mathbb{P}\{\tau < \theta-\varepsilon\} + c\,\mathbb{E}(\tau-\theta)^+,\\
R_\tau^{(\text{miss})}(\pi) &\triangleq \mathbb{E}(\theta-\tau)^+ + c\,\mathbb{E}(\tau-\theta)^+,\\
R_\tau^{(\text{exp})}(\pi) &\triangleq \mathbb{P}\{\tau < \theta\} + c\,\mathbb{E}[e^{\alpha(\tau-\theta)^+} - 1]
\end{aligned}
\right\}
$$

Standard Poisson disorder problems:

$$\mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda t}\,(\Phi_t - k)\,dt\right]$$

6. Appendix

Lebesgue decomposition of the measures. Let $\nu_0(\cdot)$ and $\nu_1(\cdot)$ be probability measures on $(\Omega, \mathcal{B}(\mathbb{R}^d))$. Then there exist a Borel function $f : \mathbb{R}^d \mapsto [0, \infty]$ and a Borel set $H \subseteq \mathbb{R}^d$ such that

$$\nu_0(H) = 0,$$
$$\nu_1(B) = \int_B f(y)\nu_0(dy) + \nu_1(B \cap H), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

If an observation Y_n falls in H, then one cannot make any error by concluding that the change from $\nu_0(\cdot)$ to $\nu_1(\cdot)$ has happened.

In general, an alarm given for the first time by the simple rule above or the decision rule obtained in the previous sections by applying to the measures $\nu_0(\cdot)$ and

$$\widetilde{\nu_1}(\cdot) = \int_{y\in\cdot} f(y)\nu_0(dy),$$

will be optimal for the linear penalty in (4).

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