Network Models

and Network Comparisons

Anna Mohr
Department of Statistics
The Ohio State University

Catherine Calder
Department of Statistics
The Ohio State University

Observational Data Reading Group
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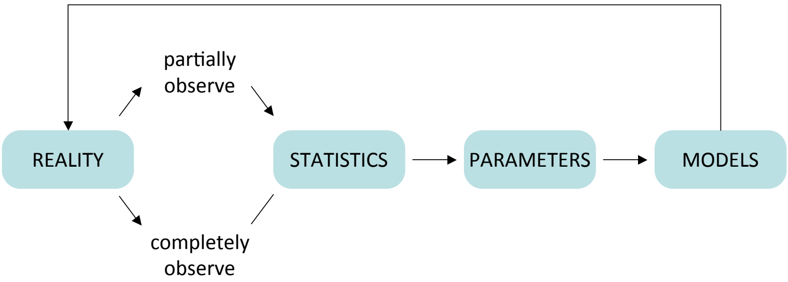
What is a model?

A Statistical Framework



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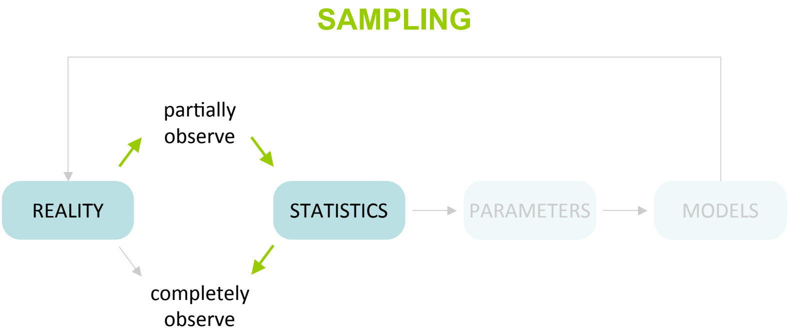

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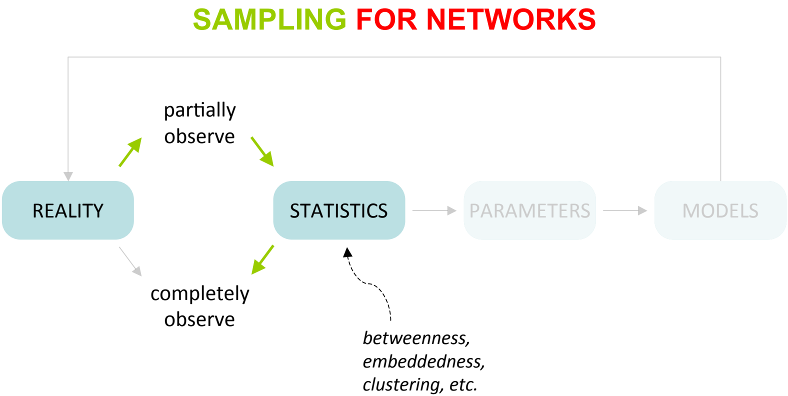

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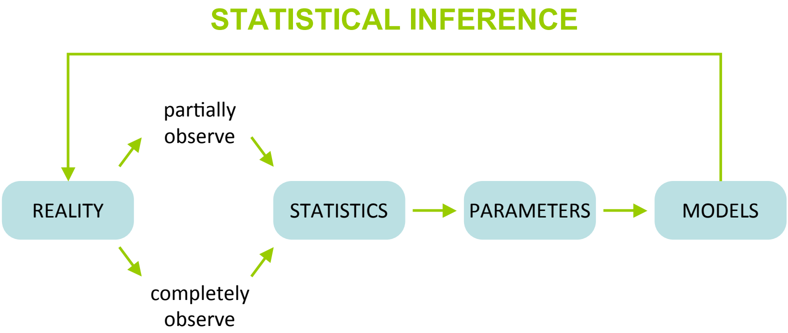

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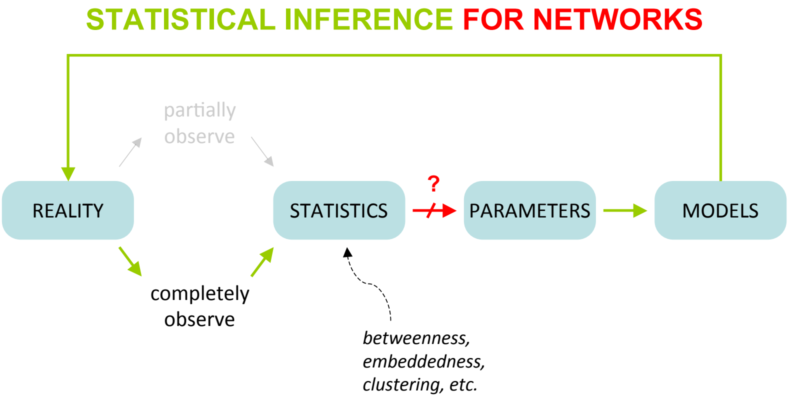

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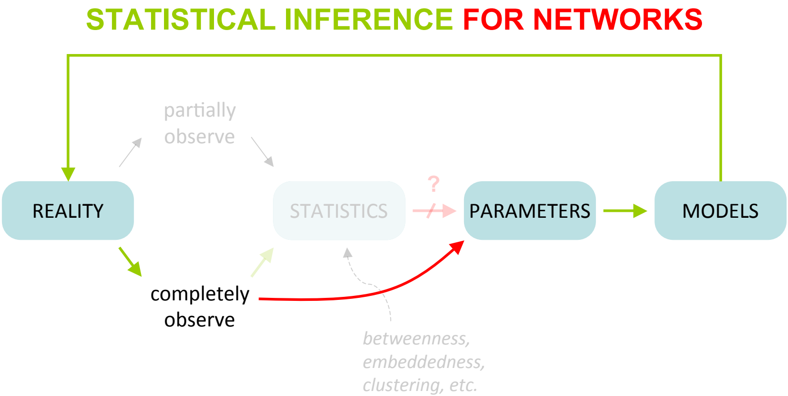

A Model for Network Graphs

a collection,

$$\{\mathbb{P}_{\boldsymbol{\theta}}(G), G \in \mathcal{G} : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$$

where $\mathcal{G}$ is a collection of possible graphs,

$\mathbb{P}_{\boldsymbol{\theta}}$ is a probability distribution on $\mathcal{G}$,

and $\boldsymbol{\theta}$ is a vector of parameters, ranging over possible values in $\boldsymbol{\Theta}$.

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Keep in mind...

Mathematical Model
- approximate relationship
- simulations

vs.

Statistical Model
- describe uncertainty
- learn about $\boldsymbol{\theta}$

A Naive Model

adjacency matrix, **Y**, for an *undirected*, *unweighted* network where each

Y_{ij} is the tie variable for vertices i and j

Logistic Regression

suppose

$$Y_{ij} \overset{iid}{\sim} \text{Bernoulli}(p)$$
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for directed graphs,

$$\mathbb{P}(Y_{ij} = y_1, Y_{ji} = y_2) \propto \exp\{y_1(\theta + \alpha_i + \beta_j) + y_2(\theta + \alpha_j + \beta_i) + y_1 y_2 \rho\}$$

Keep Improving...

take the p_1 model,

$$Y_{ij} \sim \text{Bernoulli}(p_{ij})$$
$$\text{logit}(p_{ij}) = \theta + \gamma_i + \gamma_j$$

and additionally, model

$$\boldsymbol{\gamma} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\zeta}, \quad \text{where } \zeta_i \overset{iid}{\sim} \text{Normal}(0, \sigma_\zeta^2)$$
$$\theta_{ij} = \theta + \mathbf{Z}_{ij}\delta$$

where the **X** are covariates for the set of vertices
and the **Z** are dyadic attributes

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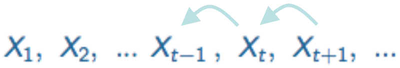
- accounts for some dependence between the Y_{ij}
- can incorporate meaningful covariates
- $\sim$ mixed effects logisitic regression

Markov Dependence

A Markov Process

let $\{X_t\}$ be a stochastic process such that

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, ...X_1 = x_1) = P(X_n = x_n | X_{n-1} = x_{n-1})$$



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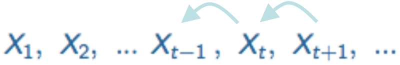
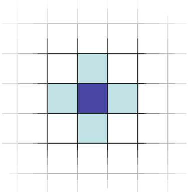


A Simple Markov Random Field

dependence on nearest neighbors



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Network Graph

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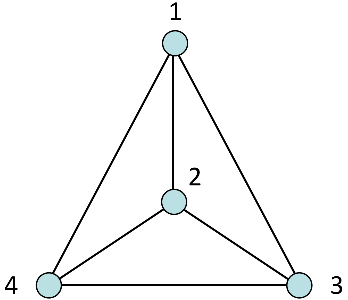

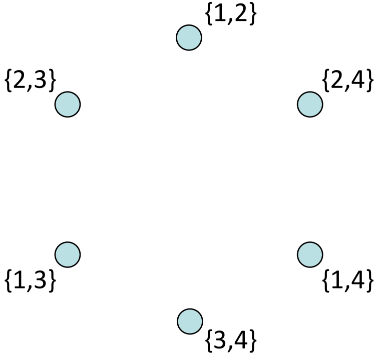

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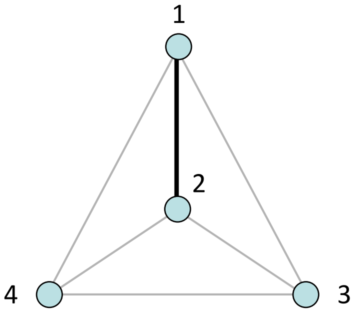

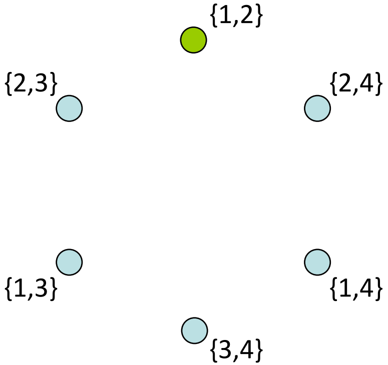

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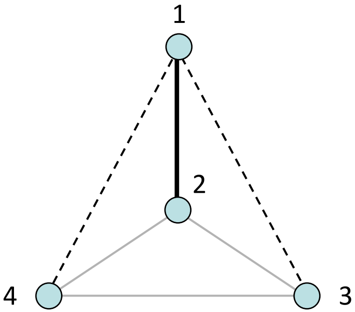

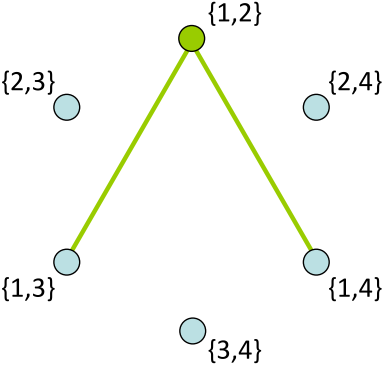

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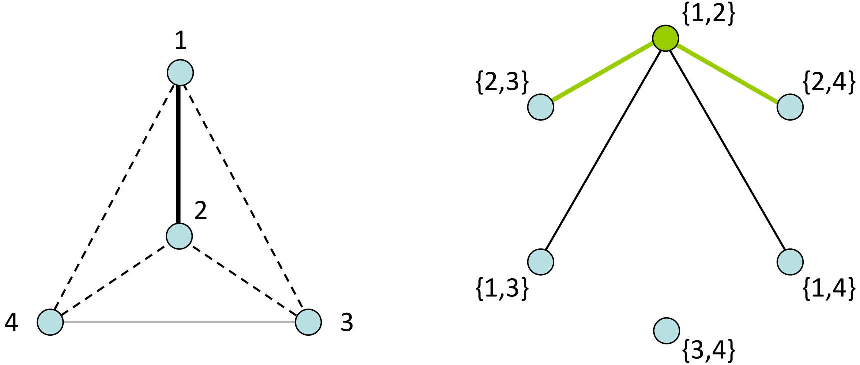

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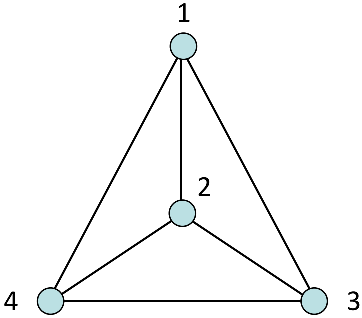

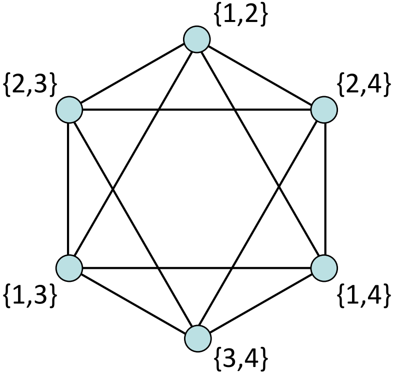

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Hammersley-Clifford theorem $\rightarrow$ any undirected graph on N_v vertices with dependence graph D has probability

$$\mathbb{P}(G) = \left(\frac{1}{c}\right) \exp\left\{\sum_{A \subseteq G} \alpha_A\right\}$$

where α_A is an indicator of the clique A in D.

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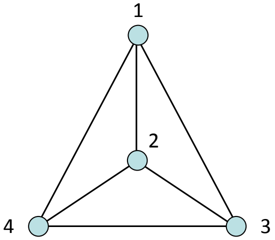

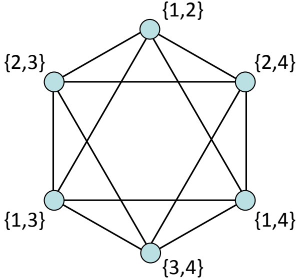

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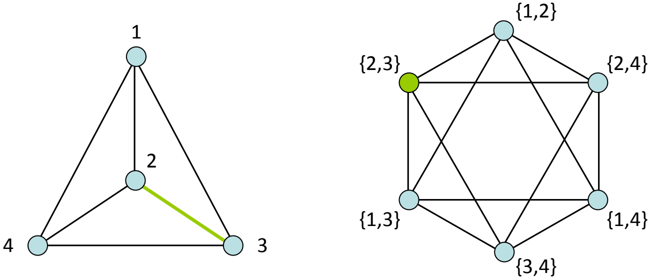

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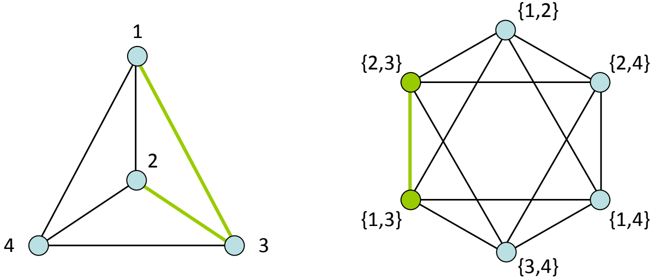

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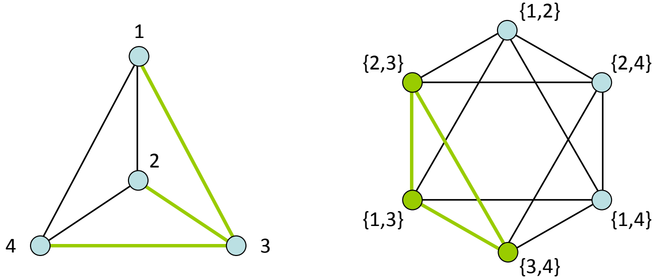

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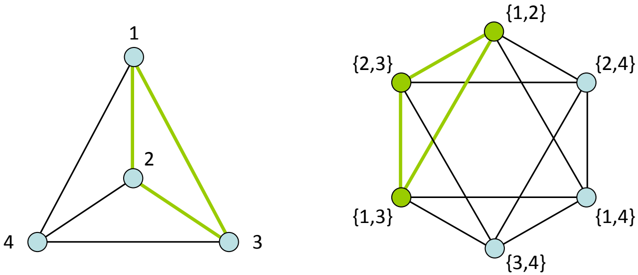

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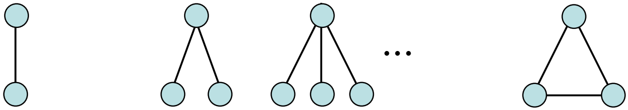

Markov Model

cliques of D are edges, k-stars, and triangles in G

Markov Model

$$\mathbb{P}_{\boldsymbol{\theta}}(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\sum_{k=1}^{N_v - 1} \theta_k S_k(\mathbf{y}) + \theta_\tau T(\mathbf{y})\right\}$$

$$\begin{aligned} \text{where } S_1(\mathbf{y}) &= N_e \\ S_k(\mathbf{y}) &= \text{ \# of k-stars} \quad \text{ for } 2 \leq k \leq N_v - 1 \\ \text{and } T(\mathbf{y}) &= \text{ \# of triangles} \end{aligned}$$

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"Triad Model"

$k \le 2$ only

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- model fitting and simulations done via MCMC
  not easy...
- model degeneracy issues
  places lots of mass on only a few outcomes
  - especially so for large N_v
  - related to the phase transitions known for the Ising model
  - change statistics for the MCMC algorithm

Exponential Random Graph Models

Exponential Family

$\mathbf{Z}$ belongs to an exponential family if its pmf can be expressed as

$$\mathbb{P}_{\boldsymbol{\theta}}(\mathbf{Z} = \mathbf{z}) = \exp\left\{\boldsymbol{\theta}' g(\mathbf{z}) - \psi(\boldsymbol{\theta})\right\}$$

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let $Y_{ij} = Y_{ji}$ be a binary r.v. indicating the presence of an edge between vertices i and j

$$\mathbb{P}_{\boldsymbol{\theta}}(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\sum_H \theta_H g_H(\mathbf{y})\right\}$$

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&= \left(\frac{e^\theta}{1 + e^\theta}\right)^{S_1(\mathbf{y})} \left(\frac{1}{1 + e^\theta}\right)^{\binom{N_v}{2} - S_1(\mathbf{y})} \\
&= \frac{\exp\left\{\theta S_1(\mathbf{y})\right\}}{(1 + e^\theta)^{\binom{N_v}{2}}}
\end{aligned}$$

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Bernoulli Model: $$\mathbb{P}_\theta(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\{\theta\, S_1(\mathbf{y})\}$$

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  complete independence

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- General Case
  ?? dependence

$$\mathbb{P}_\theta(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\sum_{H} \theta_H g_H(\mathbf{y})\right\}$$

Exponential Random Graph Models

- **Bernoulli Model**
  complete independence
  $$\mathbb{P}_\theta(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\theta S_1(\mathbf{y})\right\}$$
- **Markov Model**
  possible edges that share a vertex are dependent
  $$\mathbb{P}_\theta(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\sum_{k=1}^{N_v - 1} \theta_k S_k(\mathbf{y}) + \theta_\tau T(\mathbf{y})\right\}$$
- General Case
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  $$\mathbb{P}_\theta(\mathbf{Y} = \mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\sum_H \theta_H g_H(\mathbf{y})\right\}$$
- **Snijders et al. (2006)**

New Specifications - Snijders et al. (2006)

make use of clique-like structures...

$$\mathbb{P}_{\boldsymbol{\theta}}(\mathbf{Y}=\mathbf{y}) = \left(\frac{1}{\kappa}\right) \exp\left\{\theta_1 S_1(\mathbf{y}) + \theta_2 u^{(s)}_{\lambda_1}(\mathbf{y}) + +\theta_3 u^{(t)}_{\lambda_2}(\mathbf{y}) + \theta_4 u^{p}_{\lambda_2}(\mathbf{y})\right\}$$

where $S_1(\mathbf{y}) = N_e$

$$u^{(s)}_{\lambda}(\mathbf{y}) = \sum_{k=2}^{N_v-1} (-1)^k \frac{S_k(\mathbf{y})}{\lambda^{k-2}}$$ alternating k-stars

$$u^{(t)}_{\lambda}(\mathbf{y}) = \sum_{i<j} y_{ij} \sum_{k=1}^{N_v-2} \left(\frac{-1}{\lambda}\right)^{k-1} \binom{L_{2ij}}{k}$$ alt. k-triangles

$$u^{p}_{\lambda}(\mathbf{y}) = \lambda \sum_{i<j} \left\{1 - \left(1 - \frac{1}{\lambda}\right)^{L_{2ij}}\right\}$$ alt. independent two-paths

New Specifications - Snijders et al. (2006)

k-triangles

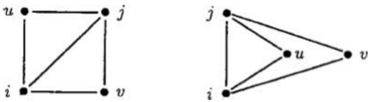

FIGURE 3. Two examples of a two-triangle.

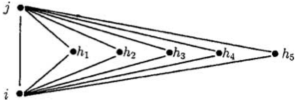

FIGURE 4. A k-triangle for $k = 5$, which is also called a five-triangle.

independent two-paths

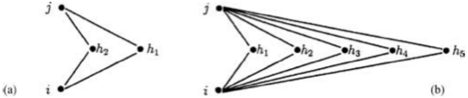

FIGURE 5. Two-independent two-paths (a) and five-independent two-paths (b).

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- fewer, less severe issues with model degeneracy
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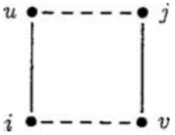
  $\rightarrow$ curved exponential family
- satisfies (weaker) **partial conditional dependence**

  Y_{iv} and Y_{uj} are conditionally dependent only if one of the two conditions hold:
  1. $\{i, v\} \cap \{u, j\} \neq \emptyset$
  2. $y_{iu} = y_{vj} = 1$



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- **Statistical Models**

  Simple Logistic Regression / Bernoulli Model

  p_1 Model

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- **Mathematical Models**

  Random Graphs – *CUG, Erdos-Renyi, Generalized*
  Small World
  Preferential Attachment

Random Graphs

a **conditional uniform graph (CUG)** distribution with sufficient statistic $\mathbf{t}$ taking on value $\mathbf{x}$:

$$\mathbb{P}(G = g|\mathbf{t}, \mathbf{x}) = \frac{1}{|\{g' \in \mathcal{G} : \mathbf{t}(g') = \mathbf{x}\}|} I_{\{g' \in \mathcal{G} : \mathbf{t}(g') = \mathbf{x}\}}(g)$$

where $\mathbf{t} = (t_1, ... t_n)$ is an n-tuple of real-valued functions on $\mathcal{G}$ and $\mathbf{x} \in \mathbb{R}^n$ is a known vector.

- pick a particular $\mathcal{G}$ and specify uniform probability

Special Cases

an **Erdos-Renyi random graph** puts uniform probablity on $\mathcal{G}_{N_v,N_e}$ so that

$$\mathbb{P}(G = g|N_v, N_e) = \frac{1}{\binom{N}{N_e}} I_{\{g \in \mathcal{G}_{N_v,N_e}\}}(g)$$

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a **generalized random graph** puts uniform probability on $\mathcal{G}_{N_v,t}$ where t is any other statistic/motif/characteristic of G.

- degree distribution $\Rightarrow N_e$ fixed

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- mathematical models
- Erdos-Renyi appears to be the most commonly used
  - most thoroughly studied
    degree distribution, probability of connectedness, etc.
  - easy to work with

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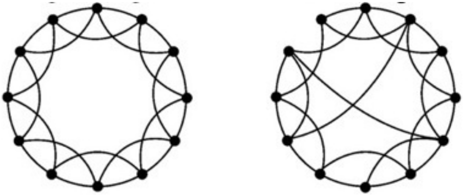
CONS
unrealistic
degree dist. is not broad enough
levels of clustering too low

Some Other Mathematical Models

Watts-Strogatz Small World Model

0. lattice of N_v vertices
1. randomly "rewire" each edge independently and with probability p, such that we change one endpoint of that edge to a different vertex (chosen uniformly)



- high levels of clustering, yet small distances between most nodes

Some Other Mathematical Models

Barabasi-Albert Preferential Attachment Model
(a network growth model)

0. $G^{(0)}$ of $N_v^{(0)}$ vertices and $N_e^{(0)}$ edges

$$\vdots$$

t. $G^{(t)}$ is created by adding a vertex of degree $m \geq 1$ to $G^{(t-1)}$, where the probability that this new vertex is connected to any existing vertex in $G^{(t-1)}$ is

$$\frac{d_v}{\sum_{v' \in V} d_v}, \quad \text{where } d_v \text{ is the degree of vertex } v$$

- can achieve broad degree distributions

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Thank you!!

Some References

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