Stat 742 Homework #7 Solution Fall 2008

6.10) Let a = 3, n = 2, and n_{ij} = 2. Then the design matrix is

$$\mathbf{Xb} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{31} \\ \gamma_{32} \end{bmatrix} = [\mathbf{X_0} \quad \mathbf{X_1}] \begin{bmatrix} \mathbf{b_0} \\ \mathbf{b_1} \end{bmatrix}$$

$\mathbf{X_0}$ and $\mathbf{b_0}$ refer to the design matrix and parameters for the overall mean and the main effects $\alpha_i, i = 1,2,3$ and $\beta_j, j = 1,2,$ and $\mathbf{X_1}$ and $\mathbf{b_1}$ for the six interaction terms. The hypotheses for no interaction are H_0: $\mathbf{b_1 = 0}$ vs. H_a: $\mathbf{b_1 \neq 0}$.

But, the γ_{ij}'s are not estimable, since their coefficient vectors are confounded with the man effects. [The columns (1, 2) (3, 4) (5, 6) in $\mathbf{X_1}$ add up to the columns for $\alpha_1(\alpha_2)(\alpha_3)$ respectively. Similarly, the columns (1,3,5) (2,4,6) in $\mathbf{X_1}$ add up to the columns for $\beta_1(\beta_2)$ respectively. However, since all six columns add up to the first column in $\mathbf{X_0}$, the rank of $(P_X - P_{X_0})$ = 2]. Now, the SSE for the full model is $\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}$

The SSE for the reduced model is $\mathbf{y}'\left(\mathbf{I} - \mathbf{P_{X_0}}\right)\mathbf{y}$

So $SSE_{Reduced} - SSE_{Full} = Q(\hat{\mathbf{b}}_\mathbf{H}) - Q(\hat{\mathbf{b}}) = \mathbf{y}'\left(\mathbf{P_X} - \mathbf{P_{X_0}}\right)\mathbf{y}$

The test stat is $F = \dfrac{\mathbf{y}'\left(\mathbf{P_X} - \mathbf{P_{X_0}}\right)\mathbf{y}/s}{\mathbf{y}'(\mathbf{I} - \mathbf{P_X})\mathbf{y}/(N-r)}$

Here, rank($\mathbf{X}$) = r =6, rank($\mathbf{X_0}$) = 4, so s = rank$(P_X - P_{X_0})$ = 2 and N – r = 12 – 6 = 6

So the test stat F is based on the $F_{2,6}$ distribution.

6.11) The design matrix is

$$\mathbf{Xb} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = [\mathbf{x_0} \ \ \mathbf{x_1}]\mathbf{b}$$

The hypothesis is H_0: $\beta_1 = 0$ vs. H_1: $\beta_1 \neq 0$

Rank($\mathbf{X}$) = 2 and Rank(x_0) = 1, so s = 2 – 1 = 1 and N – r = 5 – 2 = 3. Furthermore, since $\bar{x} = 3$, and $\sum_{i=1}^{5}(x_i - \bar{x})^2 = 10$, it follows that $\hat{\beta}_0 = \bar{Y} - 3\hat{\beta}_1$ and $\hat{\beta}_1 = \sum_{i=1}^{5}(i-3)Y_i/10$.

So the test stat is based on the $F_{1,3}$, or equivalently a t_3 distribution. Since, $F_{1,3,.05} = 10.128$

The test stat in this case is $F^* = \frac{\mathbf{y}'(\mathbf{P_X}-\mathbf{P_{X_0}})\mathbf{y}/1}{\mathbf{y}'(\mathbf{I}-\mathbf{P_X})\mathbf{y}/3} = \frac{10\hat{\beta}_1^2}{\frac{1}{3}[\sum_{i=1}^{5}(Y_i-\bar{Y})^2 - 10\hat{\beta}_1^2]}$.

Of course, H_0: $\beta_1 = 0$ is rejected if $F^* > 10.128$.

If $\beta_1 = \delta$, then the distribution of the test stat F* is a non-central F with 1 and 3 d.f., and non-centrality parameter $\varphi = ½\ 10\delta^2/\sigma^2$, i.e., $F_{1,3}(\varphi = ½\ 10\delta^2/\sigma^2)$. Thus,

If $\beta_1 = 0.1$, then $F^* \sim F_{1,3}(\varphi = 5(.01)^2/\sigma^2)$, power = $P(F_{1,3}(5(.01)^2/\sigma^2) > 10.128)$
If $\beta_1 = 0.2$, then $F^* \sim F_{1,3}(\varphi = 5(.02)^2/\sigma^2)$, power = $P(F_{1,3}(5(.02)^2/\sigma^2) > 10.128)$
If $\beta_1 = 0.3$, then $F^* \sim F_{1,3}(\varphi = 5(.03)^2/\sigma^2)$, power = $P(F_{1,3}(5(.03)^2/\sigma^2) > 10.128)$.

6.18) In this case, $n_i = n = 5$, and $a = 3$, so there are a total of $\binom{3}{2} = 3$ $\alpha_i - \alpha_j$ pairs. Note that, since $E(\hat{\sigma}^2) = \sigma^2$, but $E(\hat{\sigma}) \neq \sigma$, it will be easier to find Expected squared length of the intervals. Furthermore, each of these intervals is of the type $\hat{\theta} \pm d\ \hat{\sigma}$, where d changes from procedure to procedure, with the length = $2d\hat{\sigma}$. Therefore, E[(length)^2] = $4d^2\sigma^2$.

Bonferroni Intervals: $d_B = \left(t_{12,(\frac{\alpha}{6})}\sqrt{\frac{2}{5}}\right)$

Scheffe: $d_S = \left(\sqrt{\frac{2}{5}}\sqrt{2F_{2,12,\alpha}}\right)$

Tukey: $d_T = \left(\frac{1}{\sqrt{5}}q_{3,12,\alpha}\right)$ [**The book has a typo for Tukey d.f. : n(a-1) should be (N-a).]**

Using the Tables for t, F and q distribution, the values of $\bar{d}[=(d/\sqrt{2/5})]$ for $\alpha = 0.1$, are

$\bar{d}_B \approx 2.4,\ \bar{d}_S = 2.37,\ \bar{d}_T =)$

6.24) The design matrix is $\mathbf{Xb} = \begin{bmatrix} 1_{n_1} & 1_{n_1} & 0 & \dots & 0 \\ 1_{n_2} & 0 & 1_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{n_a} & 0 & 0 & \dots & 1_{n_a} \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix}$

a) $\mathbf{P'b} = [0 \quad n_1 \quad \dots \quad n_a] \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix}$

b) If **P'b** were estimable, **P'** would be in the C(**X'**). However, **P** cannot be obtained from any linear combination of the columns of **X'**, since, the only way to get n_i to be the coefficients of α_i is to consider $n_1x_1 + n_2x_2 + \dots + n_ax_a$, but the first element will not be 0. So **P'** is not in C(**X'**), and thus **P'b** is not estimable.

c) $\mathbf{X'Xb} = \begin{bmatrix} N & n_1 & n_2 & \dots & n_a \\ n_1 & n_1 & 0 & \dots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_a & 0 & 0 & \dots & n_a \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{X'y} = \begin{bmatrix} N\bar{y}_{..} \\ n_1\bar{y}_{1.} \\ \vdots \\ n_a\bar{y}_{a.} \end{bmatrix}$ for $N = \Sigma n_i$

Taking the g-inverse of (X'X) using the a by a submatrix on the lower right:

$\hat{\mathbf{b}} = \mathbf{(X'X)^{-}X'y} = \begin{bmatrix} 0 & 0 \\ 0 & \text{diag}(n_i^{-1}) \end{bmatrix} \begin{bmatrix} N\bar{y}_{..} \\ n_1\bar{y}_{..} \\ \vdots \\ n_a\bar{y}_{..} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{y}_{1.} \\ \vdots \\ \bar{y}_{a.} \end{bmatrix}$, and this solution is not unique.

Now the general solution is $\hat{\mathbf{b}} + \mathbf{(I - (X'X)^{-}X'X)}z = \begin{bmatrix} z \\ \bar{y}_{1.} - z \\ \vdots \\ \bar{y}_{a.} - z \end{bmatrix}$

d) $\mathbf{P'\hat{b}} = \sum_{i=1}^{a} n_i(\bar{y}_{i.} - z) = 0 \Rightarrow z\sum_{i=1}^{a} n_i = \sum_{i=1}^{a} n_i\bar{y}_{i.} \Rightarrow z = \sum_{i=1}^{a} n_i\bar{y}_{i.} / \sum_{i=1}^{a} n_i$

So a solution that satisfies the constraint is $\hat{\mathbf{b}} = \begin{bmatrix} 0 \\ \bar{y}_{1.} \\ \vdots \\ \bar{y}_{a.} \end{bmatrix} + \frac{\sum_{i=1}^{a} n_i\bar{y}_{i.}}{\sum_{i=1}^{a} n_i} \begin{bmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$.

[Note that this solution is same as if we substitute the side condition, $\sum n_i\alpha_i = 0$, in the normal equations. Then the normal equations are $N\mu = N\bar{y}_{..}$, $n_i(\mu+\alpha_i) = y_{i.}, i = 1, \cdots, a.$

e) $\begin{bmatrix} \mathbf{X'X} & \mathbf{P} \\ \mathbf{P'} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{b}} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{X'X\hat{b}} + \mathbf{P\theta} \\ \mathbf{P'\hat{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{X'y} + \mathbf{P\theta} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{X'y} \\ \mathbf{0} \end{bmatrix}$ iff $\theta = 0$

This is true since $\hat{\mathbf{b}}$ satisfies the normal equation and $\mathbf{P'\hat{b}} = \mathbf{0}$ as shown in part d.

f) $\begin{bmatrix} \mathbf{X'X} & \mathbf{P} \\ \mathbf{P'} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} N & n_1 & n_2 & \dots & n_a & 0 \\ n_1 & n_1 & 0 & \dots & 0 & n_1 \\ n_2 & 0 & n_2 & \cdots & 0 & n_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n_a & 0 & 0 & \dots & n_a & n_a \\ 0 & n_1 & n_2 & \dots & n_a & 0 \end{bmatrix} \xrightarrow[\text{(Col 1}-\text{Col a+2)}]{} \begin{bmatrix} N & n_1 & n_2 & \dots & n_a & 0 \\ 0 & n_1 & 0 & \dots & 0 & n_1 \\ 0 & 0 & n_2 & \cdots & 0 & n_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n_a & n_a \\ 0 & n_1 & n_2 & \dots & n_a & 0 \end{bmatrix}$

$\xrightarrow[\text{Col a+2}-\text{sum rows 2 through a+1}]{} \begin{bmatrix} N & n_1 & n_2 & \dots & n_a & 0 \\ 0 & n_1 & 0 & \dots & 0 & n_1 \\ 0 & 0 & n_2 & \cdots & 0 & n_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n_a & n_a \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$

Now the reduced matrix is a triangular matrix and thus is nonsingular.

g) **(X'X + PP')b = X'Xb + PP'b = X'y + P**(0) = **X'y**

h) $\begin{bmatrix} \mathbf{X} \\ \mathbf{P'} \end{bmatrix} = \begin{bmatrix} 1_{n_1} & 1_{n_1} & 0 & \ldots & 0 \\ 1_{n_2} & 0 & 1_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{n_a} & 0 & 0 & \ldots & 1_{n_a} \\ 0 & n_1 & n_2 & \ldots & n_a \end{bmatrix}$

X has rank a and normally, the sum of rows 2 through a+1 add up to row 1, but in this case, it does not since $\Sigma n_i \neq 0$. So this matrix is nonsingular.

7.9) If we consider the full model,

$$\mathbf{Xb} = \begin{bmatrix} 1_{n_1} & x_1 1_{n_1} & x_1^2 1_{n_1} & \ldots & x_1^{a-1} 1_{n_1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1_{n_a} & x_a 1_{n_a} & x_a^2 1_{n_a} & \ldots & x_a^{a-1} 1_{n_a} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{a-1} \end{bmatrix} = [X_0 \ \ X_1] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix},$$

and x_i's are all different, the column rank of the design matrix is full (= a) since no column of X can be written as a linear combination of other columns.

To test H_0: $\beta_2 = \ldots = \beta_{a-1} = 0$, *i.e.*, $[b_1 = 0]$, in this model, the reduced design matrix for the simple linear regression model is X_0.

For the lack of fit test, the full ANOVA model is given by equation (7.10),

$$E(y) = Z\mu = \begin{bmatrix} 1_{n_1} & 0 & \ldots & 0 \\ 0 & 1_{n_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1_{n_a} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_a \end{bmatrix}.$$

The column rank of this matrix is also a, and both β and μ are a-dimensional vectors. Let z_i denote the ith column of the matrix Z. Then $\frac{1}{\sqrt{n_i}} z_i, i = 1, \cdots, a,$ are mutually orthogonal vectors, each of length 1, so they form an ortho-normal basis of the column space of Z. Clearly, the first column of $X_0 = \sum z_i$, and the second column of $X_0 = \sum x_i z_i$. Therefore, $C(X_0) \subset C(Z)$. More generally, the kth column of $X = \sum x_i^{k-1} z_i, k = 1, 2, \cdots, a,$ therefore, $C(X) \subset C(Z)$. However, since rank(X) = rank(Z) = a, it follows that $C(X) = C(Z)$. Hence the full ANOVA model and the polynomial model are equivalent to each other. Furthermore, since the reduced model under H, is nested in the polynomial model, testing H is equivalent to testing for the lack of fit of the simple linear regression.

7.12) a) The row-reduced matrix is $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

=> $x_1 + x_4 + x_8 = 0$, $x_2 - x_4 + x_7 - x_8 = 0$, $x_3 - x_4 + x_7 - x_8 = 0$, $x_5 - x_8 = 0$, $x_6 - x_7 = 0$.

=> $x_1 = -x_4 - x_8$, $x_2 = x_4 - x_7 + x_8$, $x_3 = x_4 - x_7 + x_8$, $x_5 = x_8$, and $x_6 = x_7$

So $N(X) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{v_1, v_2, v_3\}$

b) To be estimable, the vectors must be orthogonal to all vectors in the null space.

i) $\beta_1 - \beta_3 = [0\ 0\ 0\ 0\ 1\ 0\ \text{-}1\ 0]b = \lambda'b \Rightarrow \lambda'v_1 = 0, \lambda'v_2 = -1, \lambda'v_3 = 1$; not estimable

ii) $\alpha_2 - \alpha_3 = [0\ 0\ 1\ \text{-}1\ 0\ 0\ 0\ 0]b \Rightarrow \lambda'v_1 = 0, \lambda'v_2 = -1, \lambda'v_3 = 1$; not estimable

iii) $\beta_1 - \beta_4 = [0\ 0\ 0\ 0\ 1\ 0\ 0\ \text{-}1]b \Rightarrow \lambda'v_1 = 0; \lambda'v_2 = 0, \lambda'v_3 = 0$; estimable

iv) $\mu + \alpha_1 = [1\ 1\ 0\ 0\ 0\ 0\ 0\ 0]b \Rightarrow \lambda'v_1 = 0, \lambda'v_2 = -1, \lambda'v_3 = 0$; not estimable.

For this simple model, estimability of any linear function, can be checked by writing the expected value of an arbitrary linear combinations of the observations. Now

$$E[l'Y] = l'X\beta = (\sum l_i)\mu + (l_1 + l_2)\alpha_1 + (l_3 + l_4)\alpha_2 + (l_5 + l_6)\alpha_3 + l_5\beta_1 + (l_1 + l_3)\beta_2 + (l_2 + l_4)\beta_3 + l_6\beta_4$$

Now, consider, e.g., $\beta_1 - \beta_3$, and match the coefficients in the above functions, we need $l_5 = 1, l_2 + l_4 = -1,$ and all other coefficients to be zero. This is not possible, since $l_5 + l_6 = 0, l_6 = 0 \Rightarrow l_5 = 0,$ which is a contradiction to $l_5 = 1.$ One can check that only $\beta_1 - \beta_4$ is estimable in the above list. In addition, $\beta_2 - \beta_3$ is also estimable.

c) To test H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4$, all pair-wise differences of $\beta_j's$ must be estimable. However, only $\beta_1 - \beta_4$ and $\beta_2 - \beta_3$ are estimable, and $\beta_1 - \beta_2$, $\beta_3 - \beta_4$, $\beta_1 - \beta_3$, and $\beta_2 - \beta_4$ are not estimable. So this is not a testable hypothesis.

d) $E(\mathbf{z}) = \begin{bmatrix} E(y_{12}) - E(y_{13}) \\ E(y_{22}) - E(y_{23}) \\ E(y_{31}) - E(y_{34}) \end{bmatrix} = \begin{bmatrix} \beta_2 - \beta_3 \\ \beta_2 - \beta_3 \\ \beta_1 - \beta_4 \end{bmatrix}$

$V(z_1) = V(y_{12}) + V(y_{13}) - 2cov(y_{12}, y_{13}) = 2\sigma^2$ since all y_{ij} are independent and $V(y_{ij}) = \sigma^2$

Likewise, $V(z_2) = V(z_3) = 2\sigma^2$.

$Cov(z_1, z_2) = cov(y_{12}, y_{22}) - cov(y_{12}, y_{23}) - cov(y_{13}, y_{22}) + cov(y_{13}, y_{23}) = 0$

$Cov(z_1, z_3) = cov(y_{12}, y_{31}) - cov(y_{12}, y_{34}) - cov(y_{13}, y_{31}) + cov(y_{13}, y_{34}) = 0$

Similarly, $Cov(z_2, z_3) = 0$
So $cov(\mathbf{z}) = 2\sigma^2\mathbf{I_3}$, i.e., z_1, z_2, and z_3 are mutually independent random variables.

e) $E(z_1 - z_2) = 0$ and $V(z_1 - z_2) = V(z_1) + V(z_2) = 4\sigma^2$
So $z_1 - z_2 \sim N(0, 4\sigma^2) \Rightarrow [1/(2\sigma)](z_1 - z_2) \sim N(0, 1) \Rightarrow [1/(4\sigma^2)](z_1 - z_2)^2 \sim \chi_1^2$
where $c_1 = 1/(4\sigma^2)$, whether or not H is true.
f) Now, $\frac{z_1+z_2}{\sqrt{2}} \sim N\left(\sqrt{2}(\beta_2 - \beta_3), 2\sigma^2\right)$. Since, z_1, z_2 are iid normal random variables, $z_1 + z_2$ and $z_1 - z_2$ are independently distributed. In addition, $\beta_2 - \beta_3 = 0 \Rightarrow \frac{z_1+z_2}{2\sigma} \sim N(0,1)$.
Furthermore, $z_3 \sim N(\beta_1 - \beta_4, 2\sigma^2)$, independent of $z_1 + z_2$. In addition, $\beta_1 - \beta_4 = 0$
$\Rightarrow \frac{z_3}{\sqrt{2}\sigma} \sim N(0,1)$.
Therefore, $\frac{1}{2\sigma^2}\left(\frac{(z_1+z_2)^2}{2} + {z_3}^2\right) \sim \chi_2^2(\varphi)$, where $\varphi = \frac{1}{2*2\sigma^2}\left(2(\beta_2 - \beta_3)^2 + (\beta_1 - \beta_4)^2\right)$,
and it is independent of $z_1 - z_2$. Of course, H implies $\varphi = 0$, but φ is zero iff $\beta_2 = \beta_3$ and $\beta_1 = \beta_4$.

g) As shown in part e, the denominator of G is distributed as χ_1^2, and as shown in part f, the numerator of G is distributed as χ_2^2 under H, and they are independently distributed, hence, under H, $G \sim F_{2,1}$. From the tables, $P(F_{2,1} > 199.5|H) = 0.05$, so $c_3 = 199.5$

h) As discussed in part f) above, let $\beta_1 = \beta_4 = a$ and $\beta_2 = \beta_3 = b$, where $a \neq b$. Although H is no longer true, $\frac{1}{2\sigma^2}\left(\frac{(z_1+z_2)^2}{2} + {z_3}^2\right)$ and $[1/(4\sigma^2)](z_1 - z_2)^2$ are still independent random variables each with a χ^2 distribution. So G still has the same $F_{2,1}$ distribution, thus $P(G > c_3|\beta^*) = 0.05$.