Stat 742 **Homework 5 Solutions** **Fall 2008**

5.3) a) $E_Y(e^{tY}) = E_X[E_{Y|X}(e^{tY})] = E_X\left(e^{(a+bx)t+\frac{1}{2}\gamma^2t^2}\right) = e^{at+\frac{1}{2}\gamma^2t^2}M_x(bt)$

$= e^{at+\frac{1}{2}\gamma^2t^2}e^{\mu bt+\frac{1}{2}\sigma^2b^2t^2} = e^{(a+\mu b)t+\frac{1}{2}(\gamma^2+\sigma^2b^2)t^2}$

Thus, the mgf of Y is that of a $N(a+b\mu, \gamma^2 + \sigma^2b^2)$ distribution. It follows from the Uniqueness Theorem that $Y \sim N(a+b\mu, \gamma^2 + \sigma^2b^2)$.

b) $E_Y(e^{tY}) = E(e^{t(a+bX+U)}) = e^{at}M_X(bt)M_U(t)$

$= e^{at}e^{\mu bt+\frac{1}{2}\sigma^2b^2t^2}e^{\frac{1}{2}\gamma^2t^2} = e^{(a+\mu b)t+\frac{1}{2}(b^2\sigma^2+\gamma^2)t^2}$

Thus, the mgf of Y is that of a $N(a+b\mu, \gamma^2+b^2\sigma^2)$. So $Y \sim N(a+b\mu, \gamma^2+b^2\sigma^2)$.

c) $f_{XY} = f_{Y|X}f_X = \lambda e^{-\lambda x}xe^{-xy} = \lambda xe^{-(\lambda+y)x}$

$f_Y = \int_0^\infty f_{XY}\,dx = \int_0^\infty \lambda xe^{-(\lambda+y)x}dx = \lambda\int xe^{-(\lambda+y)x}dx$

The integral is a Gamma function with shape parameter 2 and scale parameter $(\lambda+y)$.

Therefore, $f_Y(y) = \frac{\lambda}{(\lambda+y)^2} I(y > 0)$.

d) Suppose $X \sim Exp(\lambda)$ and $U \sim Exp(\gamma)$

$f_Y(y) = \int_0^\infty f_{X,Y-X}\,dx = \int_0^\infty \lambda\gamma e^{-\lambda x-\gamma(y-x)}dx = \frac{\lambda\gamma e^{-\gamma y}}{(\lambda-\gamma)}$. We want to find γ such that $\frac{\lambda\gamma e^{-\gamma y}}{(\lambda-\gamma)} = \frac{\lambda}{(\lambda+y)^2}$, for all y. Since the function $(\lambda+y)^2e^{-\gamma y}$ is not a constant (the derivative of its log $= \frac{2}{\lambda+y} - \gamma$, which is not zero for all y), so no such value of γ exists.

e) Answers will vary depending on distribution used. One such example is the uniform distribution. Let $X \sim Unif(0,1)$ and $Y|X \sim Unif(x, x+1)$.

$f_Y = \int_0^1 f_{Y|X}f_X\,dx = 1I(0 < x < 1, x < y < x+1)$, call this result *

Now suppose $Y = X + U$ where $U \sim Unif(0,a)$

$f_Y(y) = \int_0^1 (\frac{1}{a})dx = \frac{1}{a} I(0 < x < 1, x < y < x + a)$, call this result **

So if a = 1 such that $U \sim Unif(0,1)$, then the results * and ** are equal.

5.11) From result 5.6, $E(U) = p + 2\varphi$ and $V(U) = 2p + 8\varphi$.

The mgf of U is $m_U(t) = (1 - 2t)^{-p/2}e^{2\varphi t/(1-2t)}$

$m_U'(t) = p(1 - 2t)^{-(p+2)/2}e^{2\varphi t/(1-2t)} + (1 - 2t)^{-p/2}e^{2\varphi t/(1-2t)}2\varphi\left[\frac{(1-2t)+2t}{(1-2t)^2}\right]$

$= p(1 - 2t)^{-(p+2)/2}e^{2\varphi t/(1-2t)} + (1 - 2t)^{-(p+4)/2} e^{2\varphi t/(1-2t)}2\varphi$

$m_U'(0) = p + 2\varphi$, thus $E(U) = p + 2\varphi$

$m_U''(t) = p(p+2)(1-2t)^{-(p+4)/2}e^{2\varphi t/(1-2t)} + p(1-2t)^{-(p+2)/2}\, e^{2\varphi t/(1-2t)} 2\varphi\left[\frac{1}{(1-2t)^2}\right]$

$+ (p+4)(1-2t)^{-(p+6)/2}e^{2\varphi t/(1-2t)}2\varphi + (1-2t)^{-(p+4)/2}e^{2\varphi t/(1-2t)}4\varphi^2\left[\frac{1}{(1-2t)^2}\right]$

$m_U''(0) = p(p+2) + 2\varphi p + 2\varphi(p+4) + 4\varphi^2 = E(U^2)$

So $V(U) = E(U^2) - E(U)^2 = \sout{p^2} + 2p + \sout{2\varphi p + 2\varphi p} + 8\varphi + \sout{4\varphi^2} - (\sout{p^2} + \sout{4\varphi p} + \sout{4\varphi^2})$

$\Rightarrow V(U) = 2p + 8\varphi$.

A simpler approach is to use the cumulant generating function $g_U(t) = \ln(m_U(t))$. Note that, $E(U) = g'(0)$, $V(U) = g''(0)$. Now,

$g_U(t) = \ln\left(m_U(t)\right) = -\frac{p}{2}\ln(1-2t) + \frac{2\varphi t}{1-2t}$. Therefore, $g_U'(t) = \frac{p+2\varphi}{1-2t} + \frac{4\phi t}{(1-2t)^2}$, and

$g_U''(t) = 2\frac{p+2\varphi}{(1-2t)^2} + \frac{4\phi}{(1-2t)^2} + \frac{16\phi t}{(1-2t)^3}$. Hence, $g_U'(0) = p + 2\varphi$, and $g_U''(0) = 2p + 8\varphi$.

Now, Lemma 4.1 states that if $E(Z) = \mu$ and $Cov(Z) = \Sigma$, then $E(Z'AZ) = \mu'A\mu + trace(A\Sigma)$. Suppose $Z \sim N(\mu, I)$.

Using the Result 5.10, $A = I$ such that $U = X'X \sim \chi^2_p(\varphi = \mu'\mu/2)$

Then $E(U) = E(X'X) = p + \mu'\mu = p + 2\varphi$. Note that $trace(A\Sigma) = trace(I) = p$.

Result 4.6 states that if P is symmetric and e is a vector such that

$E(e_i) = 0$, $V(e_i) = \sigma^2$, $E(e_i^3) = \gamma_3$, and $E(e_i^4) = \gamma_4$, then

$V((\mu+e)'P(\mu+e)) = 4\sigma^2(\mu'P^2\mu) + 4\gamma_3\Sigma_i\mu_iP_{ii}\Sigma_jP_{ji} + 2\sigma^4\Sigma_{i\neq j}P_{ij}^2 + \Sigma_i(\gamma_4 - \sigma^4)P_{ii}^2$.

Here use $\sigma^2 = 1$, $\gamma_3 = 0$, and $\gamma_4 = 3$.

Then $V((z-\mu)'P(z-\mu)) = 4(\mu'\mu) + 2\Sigma_{i\neq j}I_{ij}^2 + 2\Sigma_iI_{ii}^2 = 8(\mu'\mu/2) + 2*0 + 2p = 8\varphi + 2p$

5.14) a) From 5.12, $m_U(t) = |I - 2tAV|^{-1/2}\exp\{-(1/2)[\mu'V^{-1}\mu - \mu'(V-2tVAV)^{-1}\mu]\}$

Now, if $g_U(t) = \log(m_U(t))$ denotes the cumulant generating function, then

$E(U) = g'(0)$, $V(U) = g''(0)$.

$g_U(t) = -(1/2)\log(|I - 2tAV|) - (1/2)[\mu'V^{-1}\mu - \mu'(V-2tVAV)^{-1}\mu]$

Recall from problem A.73 on homework 1 that the derivative of $|I + tV|$ at t=0 is trace(V).

$g_U'(t) = -\frac{1}{2}\frac{\text{trace }(-2AV)}{|I-2tAV|} + \frac{1}{2}\left[\mu'(V-2tVAV)^{-1}VAV(V-2tVAV)^{-1}\mu\right](-2)$

$= \frac{\text{trace }(AV)}{|I-2tAV|} - \left[\mu'(V-2tVAV)^{-1}(VAV)(V-2tVAV)^{-1}\mu\right]$

$g_U''(t) = 2|I - 2tAV|^{-2}trace(AV)^2 +$

$(1/2)\mu'(V-2tVAV)^{-1}(-2VAV)(V-2tVAV)^{-1}(-2VAV)(V-2TVAV)^{-1}\mu +$

$(1/2)\mu'(V-2tVAV)^{-1}(-2VAV)(V-2tVAV)^{-1}\mu$

$g_U''(t) = 2trace(AV)|I|^{-2} + 2\mu'V^{-1}(VAV)V^{-1}(VAV)V^{-1}\mu + 2\mu'V^{-1}(VAV)V^{-1}\mu$

$= 2trace(AV)^2 + 4\mu'AVA\mu = V(X'AX)$

b) If $X \sim N_p(0, V)$ then $m_U(t) = |I - 2tAV|^{-1/2} \Rightarrow g_U(t) = -(1/2)\log|I - 2tAV|$

$g_U'(t)$ = -(1/2)|I – 2tAV|$^{-1}$trace(-2AV) = trace(AV)|I – 2tAV|$^{-1}$

$g_U''(0)$ = -trace(AV)|I|$^{-2}$(trace(-2AV)) = 2trace(AV)^2 = V(X'AX)

5.18) a) (=>) Suppose A is idempotent. Then (I-A) is also idempotent. It is known that the rank of an idempotent matrix is equal to its trace. Thus rank(I-A) = p - trace(A) =p – rank(A).
=> rank(A) + rank(I – A) = p
(<=) Suppose rank(A) + rank(I – A) = p
Since A is symmetric, N(A) is orthogonal to C(A') and C(A)
=> I – A is in N(A) since rank(A) + rank(I – A) = p
=> A(I – A) = A – AA = 0 => A = AA, thus A is idempotent.
b) $(A+B)^2 = A^2 + AB + BA + B^2 = A + AB + BA + B = A + B$
=> AB + BA = 0 => AABB + ABAB = AB + ABAB => AB(I + AB) = 0
=> AB = 0
c) Note that $C(\Sigma A_i) \subseteq \Sigma C(A_i)$ => $\dim(C(\Sigma A_i)) \leq \dim(\Sigma C(A_i))$
=> $\text{rank}(\Sigma A_i) \leq \Sigma \text{rank}(A_i)$

5.21) Given X|Z and Y|Z are independent and the distribution of Y|Z=z does not depend on z
Then $f_{X,Y|Z} = f_{X|Z}f_{Y|Z} = f_{X|Z}f_Y$
=> $f_{X,Y} = \int f_{X|Z}f_{Y|Z}f_Z dz = f_Y \int f_{X|Z}f_Z dz$ since Y does not depend on Z
=> $f_{X,Y} = f_Y \int f_{X,Z} dz = f_X f_Y$
Thus, X and Y are independent random variables.

4.6) a) $E(\tilde{b}_1) = \Sigma(z_i - \bar{z})(E(y_i) - E(\bar{y}))/\Sigma(z_i - \bar{z})(x_i - \bar{x})$
$= \Sigma(z_i - \bar{z})(\beta_0 + \beta_1 x_i - 0)/ \Sigma(z_i - \bar{z})(x_i - \bar{x})$
$= [\Sigma\beta_0(z_i - \bar{z}) + \Sigma\beta_1 x_i(z_i - \bar{z})]/\Sigma(z_i - \bar{z})x_i = \beta_1$ (Since $\Sigma(z_i - \bar{z}) = 0$)
Thus, it is unbiased.
b) $V(\tilde{b}_1) = V(\Sigma(z_i - \bar{z})y_i/\Sigma(z_i - \bar{z})x_i) = \frac{\Sigma(z_i - \bar{z})^2}{[\Sigma(z_i - \bar{z})x_i]^2}\sigma^2 = \frac{\Sigma(z_i - \bar{z})^2}{[\Sigma(z_i - \bar{z})(x_i - \bar{x})]^2}\sigma^2$
c) When $x_i = z_i$, then $V(\hat{b}_1) = \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}$
So $\frac{V(\hat{b}_1)}{V(\tilde{b}_1)} = \frac{\sigma^2/\Sigma(x_i - \bar{x})^2}{\sigma^2\Sigma(z_i - \bar{z})^2/[\Sigma(z_i - \bar{z})(x_i - \bar{x})]^2} = \frac{[\Sigma(z_i - \bar{z})(x_i - \bar{x})]^2}{\Sigma(x_i - \bar{x})^2\Sigma(z_i - \bar{z})^2} \leq 1$
Thus, $\hat{b}_1$ is the BLUE.

5.26) a) $X(Z'X)^{-1}Z'X = X$, thus $(Z'X)^{-1}Z$ is a generalized inverse of X.
b) $\tilde{b} = (Z'X)^{-1}Z'y$ since (Z'X) is full rank
$\tilde{b}$ is normal with mean $E(\tilde{b}) = (Z'X)^{-1}Z'E(y) = (Z'X)^{-1}Z'Xb = b$
Variance $V(\tilde{b}) = (Z'X)^{-1}Z'V(y)Z(Z'X)^{-1} = \sigma^2(Z'X)^{-1}Z'Z(Z'X)^{-1}$
So $\tilde{b} \sim N(b, \sigma^2(Z'X)^{-1}Z'Z(Z'X)^{-1})$
c) Since $Y \sim N(Xb, \sigma^2 I)$ and $I - P_Z$ is idempotent with rank N – 2,

Then $\frac{1}{\sigma^2} y'(I - P_Z)y \sim \chi^2_{N-2}(\varphi = \frac{1}{2\sigma^2}(Xb)'(I - P_Z)(Xb))$

d) If $\beta_1 = 0$, then $\hat{\beta}_0 = \bar{y}$ and thus, $b \in C(Z')$ => $b(I - P_z) = 0$

Thus, $\varphi = (Xb)'(I - P_Z)(Xb) = 0$.

e) Given $y \sim N(Xb, \sigma^2 I)$ and $\tilde{b} = (Z'X)^{-1}Z'y$

By Corollary 5.3, $\tilde{b}$ and $I - P_Z$ are independent if $(Z'X)^{-1}Z'(I - P_z) = 0$

$= (Z'X)^{-1}Z' - (Z'X)^{-1}Z'P_z = (Z'X)^{-1}Z' - (Z'X)^{-1}Z'Z(Z'Z)^{-1}Z'$

$= (Z'X)^{-1}Z' - (Z'X)^{-1}Z' = 0$

Thus, they are independent.

f) Since $\beta_1 = 0$, $\frac{1}{\sigma^2} y'(I - P_Z)y$ is a centralized χ^2 distribution

Also, $\tilde{\beta}_1 \sim N(0, \sigma^2 v)$ => $\frac{\tilde{\beta}_1^2}{\sigma^2 v} \sim \chi_1^2$

Thus the distribution is $F_{1, N-2}$.

g) $\tilde{e} = y - X\tilde{b} = y - X(Z'X)^{-1}Z'y = (I - X(Z'X)^{-1}Z')y$ => $P = (I - X(Z'X)^{-1}Z')$

h) $P' = I - Z(Z'X)^{-1}X' \neq I - X(Z'X)^{-1}Z'$ so it is not symmetric.

$PP = I - 2X(Z'X)^{-1}Z' + X(Z'X)^{-1}Z'X(Z'X)^{-1}Z' = I - 2X(Z'X)^{-1}Z' + X(Z'X)^{-1}Z' = P$

Thus, P is idempotent.

i) $E(\tilde{e}) = Xb - X(Z'X)^{-1}Z'Xb = (I - X(Z'X)^{-1}Z')Xb = 0$

$V(\tilde{e}) = Cov(y - X\tilde{b}) = Cov((I - P)y) = \sigma^2(I - X(Z'X)^{-1}Z')(I - Z(Z'X)^{-1}X')$

Also, $\tilde{e}$ and $\tilde{b}$ are independent since $\tilde{b} = (Z'X)^{-1}Z'y$ and $\tilde{e} = (I - X(Z'X)^{-1}Z')y$

=> $(Z'X)^{-1}Z'(I - X(Z'X)^{-1}Z') = 0$ and by Corollary 5.3, they are independent.

Thus, $\begin{bmatrix}\tilde{b}\\ e\end{bmatrix} \sim N\left(\begin{bmatrix}b\\0\end{bmatrix}, \sigma^2\begin{bmatrix}(Z'X)^{-1}Z'Z(X'Z)^{-1} & 0\\ 0 & (I - X(Z'X)^{-1}Z')(I - Z(X'Z)^{-1}X')\end{bmatrix}\right)$

j) From Result 5.15, P must be symmetric and P'P must be idempotent.

$P = I - X(Z'X)^{-1}Z'$ is not symmetric as shown in part h.

In addition, P'P is not idempotent.

We cannot show that the distribution of $||e||^2/\sigma^2$ is a chi-square.

5.28) $E(X_N) = p + 2\varphi_N$ and $V(X_N) = 2p + 8\varphi_N$

Let $Z_N = \frac{X_N - (p + 2\varphi_N)}{\sqrt{2p + 8\varphi_N}}$, so that $E(Z_N) = 0$ and $V(Z_N) = 1$

The mgf of X_N is $(1 - 2t)^{p/2}e^{(\varphi t)/(1-2t)}$

The mgf of Z_N is $m_Z(t) = E[\exp\{t\frac{X_N - (p+2\varphi_N)}{\sqrt{2p+8\varphi_N}}\}] = \exp\{\frac{-(p+2\varphi_N)t}{\sqrt{2p+8\varphi_N}}\}E[\exp\{\frac{t}{\sqrt{2p+8\varphi_N}}X_N\}]$

$$= \underbrace{exp\left\{\frac{-(p+2\varphi_N)t}{\sqrt{2p+8\varphi_N}}\right\}}\underbrace{\left[1 - 2\frac{t}{\sqrt{2p+8\varphi_N}}\right]^{-p/2}}\exp\left\{\frac{2\varphi_N\left(\frac{t}{\sqrt{2p+8\varphi_N}}\right)}{1 - 2\frac{t}{\sqrt{2p+8\varphi_N}}}\right\}$$

To take the limit of ln(mgf) as $\varphi_N \to \infty$, let $t_N = \frac{t}{\sqrt{2p + 8\varphi_N}}$. Hence, $t_N \to 0$, as $\varphi_N \to \infty$.

Now, $\ln[M_Z(t)] = -(p + 2\varphi_N)t_N - \frac{p}{2}\ln(1-2t_N) + \frac{2\varphi_N t_N}{1-2t_N}$.

Since $\ln(1\text{-}x) \approx -x$, for x near zero,

$$\ln[M_Z(t)] \approx -pt_N + \frac{p}{2}2t_N - 2\varphi_N t_N + \frac{2\varphi_N t_N}{1-2t_N} = -\frac{4\varphi_N t_N^2}{1-2t_N}.$$

However, $4\varphi_N t_N^2 = 4\varphi_N \frac{t^2}{2p+8\varphi_N} \to \frac{1}{2}t^2$ as $\varphi_N \to \infty$. Therefore, $\lim_{\varphi_N \to \infty} \ln[M_Z(t)] = \frac{1}{2}t^2$.

Hence, the limiting value of $M_Z(t) = e^{\frac{1}{2}t^2}$, which is the mgf of the N(0,1) distribution. Thus, by uniqueness theorem, suitably centered and scaled X_N, converges in distribution to a normal, as $\varphi_N \to \infty$.