Problems 2.2, 2.5, 2.11, 2.15, 2.19 (2.2, 2.11, and 2.15 will be graded)

2.2) Let **u** = $\frac{\mathbf{x}}{||\mathbf{x}||} = \frac{\mathbf{x}}{\sqrt{\mathbf{x}'\mathbf{x}}}$ so that f(**u**) = **u'Au**. Note that **u'u** = 1.

Then you can write f(**u**) in Lagrangian form in example B.3: f(**u**) = **u'Au** + λ(**u'u** – 1).

Then following example B.3:

$\frac{\partial f}{\partial \boldsymbol{u}}$ = [A + A']u – 2λu = 0 => 2Au = 2λu => Au = 2λu

Thus the solution of u where $\frac{\partial f}{\partial \boldsymbol{u}} = 0$ are the normalized eigenvectors of A.

By the chain rule, $\frac{\partial f}{\partial \boldsymbol{x}} = \frac{\partial f}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = 0$ => If $\frac{\partial f}{\partial \boldsymbol{u}} = 0$, then $\frac{\partial f}{\partial \boldsymbol{x}} = 0$.

Thus, the solution to $\frac{\partial f}{\partial \boldsymbol{x}} = 0$ are the eigenvectors of **A**.

As a result, for $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ are the eigenvalues of **A**,

$\lambda_p = \max_{\boldsymbol{x}} \frac{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} \geq \frac{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} \geq \min_{\boldsymbol{x}} \frac{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} = \lambda_1$

If A is not symmetric, the same result will not hold because

$\frac{\partial}{\partial \boldsymbol{x}}[x' A x] = [A + A']x$ and therefore, the solution is not an eigenvector of A since A + A' ≠ 2A.

But, the quadratic form doesn't change if you replace A by (A + A'/2. Therefore, you can assume that A is symmetric.

2.5) Want to find **B** such that **X'XB** = **X'**.

Note that X'X is a full rank matrix in this model, so if **B** = $\mathbf{(X'X)^{-1}X'}$, then **X'XB** = $\mathbf{X'X(X'X)^{-1}X'}$ = **X'**.

B = $\mathbf{(X'X)^{-1}X'}$ =

$$\left(\begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ x_{11} & \ldots & x_{1N_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 \\ 0 & \ldots & 0 & x_{21} & \ldots & x_{2N_2} \end{bmatrix} \begin{bmatrix} 1 & x_{11} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1N_1} & 0 & 0 \\ 0 & 0 & 1 & x_{21} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & x_{2N_2} \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ x_{11} & \ldots & x_{1N_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 \\ 0 & \ldots & 0 & x_{21} & \ldots & x_{2N_2} \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & \sum x_{1i} & 0 & 0 \\ \sum x_{1i} & \sum x_{1i}^2 & 0 & 0 \\ 0 & 0 & N_2 & \sum x_{2i} \\ 0 & 0 & \sum x_{2i} & \sum x_{2i}^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 \\ x_{11} & \ldots & x_{1N_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 \\ 0 & \ldots & 0 & x_{21} & \ldots & x_{2N_2} \end{bmatrix} = \begin{bmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{bmatrix}^{-1} \begin{bmatrix} X_1' & 0 \\ 0 & X_2' \end{bmatrix}$$

Where $X_j = \begin{bmatrix} 1 & x_{j1} \\ \ldots & \ldots \\ 1 & x_{jN_j} \end{bmatrix}$. Next, apply the property of the inverse of a partitioned matrix.

$$= \begin{bmatrix} (X_1'X_1)^{-1} & 0 \\ 0 & (X_2'X_2)^{-1} \end{bmatrix} \begin{bmatrix} X_1' & 0 \\ 0 & X_2' \end{bmatrix} = (X_1'X_1)^{-1}X_1 + (X_2'X_2)^{-1}X_2$$

where $(X_j'X_j)^{-1} = \frac{1}{N_j \sum x_{ji}^2 - (\sum x_{1i})^2} \begin{bmatrix} \sum x_{ji}^2 & -\sum x_{ji} \\ -\sum x_{ji} & N_j \end{bmatrix}$

2.11) a) The rank of X is 3.

b) The dimension of C(**X**) is the rank of **X** (= 3), and there are three nonzero vectors [1,1,1,1,1]', [-1,0,0,1,1]', and [1,0,0,1,1]' .

c) The dimension of N(**X**) is the number of columns (in this case since there are fewer columns than rows) minus the dimension of C(**X**), so the dimension of N(**X**) is 1.

After doing a row reduction in solving the equation **Xx** = **0**, the equations obtained are:

$x_1 = 0$, $x_2 + x_4 = 0$, and $x_3 = 0 \Rightarrow x_4 = -x_2$, so the vector [0,-1,0,1] is in N(**X**).

d) The dimension of C(X') is also 3 and the three nonzero vectors in it are [1,-1,1,-1]', [1,1,1,1]', and [1,0,0,0]'.

e) Since the rank of X is 3, the rank of X'X is 3. Take a 3 by 3 block of X'X and then find the inverse of it. Then put 0's for all other entries.

$X'X = \begin{bmatrix} 5 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix}$ => Inverse of the 3 by 3 top-left matrix: $\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 3/8 & -1/8 \\ -1/2 & -1/8 & 7/8 \end{bmatrix}$

So a g-inverse is $\begin{bmatrix} -1/2 & 0 & -1/2 & 0 \\ 0 & 3/8 & -1/8 & 0 \\ -1/2 & -1/8 & 7/8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

f) Let **b**' = [b_1 b_2 b_3 b_4]. Calculate **Xb** = **y** and you get the following set of equations:

$b_1 - b_2 + b_3 - b_4 = y_1$

$b_1 = y_2$

$b_1 = y_3$

$b_1 + b_2 + b_3 + b_4 = y_4$

$b_1 + b_2 + b_3 + b_4 = y_5$

[1, -1, 1, -1, 1]' is not appropriate because $y_2 \neq y_3$ as one of the conditions.

For [3,1,1,2,2]' and [1,0,0,2,2]' there is a solution that satisfies all equations, so those two are appropriate response vectors.

g) For the vector [3,1,1,2,2]', you automatically get that $b_1 = 1$.

From y_4 and y_5, $b_2 + b_3 + b_4 = 1$.

From y_1, $-b_2 + b_3 - b_4 = 2$.

From adding these two equations, you get that $b_3 = 3/2$.

So from equation y_4, $b_2 + b_4 = -1/2$ and from y_1, you also get $b_2 + b_4 = -1/2$.

Thus, the solution is $b_1 = 1$, $b_3 = 3/2$, and any b_2 and b_4 that satisfies the condition $b_2 + b_4 = -1/2$.

For the vector [1,0,0,2,2]', you automatically get that $b_1 = 0$.

From y_4 and y_5, $b_2 + b_3 + b_4 = 2$

From y_1, $-b_2 + b_3 - b_4 = 1$

Adding these two equations, $b_3 = 3/2$.

So from equation y_4 after plugging in for b_1 and b_3, $b_2 + b_4 = 1/2$

Thus, the solution is $b_1 = 0$, $b_3 = 3/2$, and any b_2 and b_4 that satisfies the condition $b_2 + b_4 = 1/2$.

h) By result 2.8, If C(**X**) = C(**W**), then $\mathbf{P_X} = \mathbf{P_W}$ where $\mathbf{P_X} = \mathbf{X(X'X)^-X'}$

Using the g-inverse found in (e), $\mathbf{P_X}$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

This is equivalent to $\mathbf{P_U}$, so the column space of **U** is the same as **X**.

2.15) a) Suppose **S** and **T** are matrices such that **X = WS** and **X = WT**

=> **WS** = **WT** => **W(S – T)** = **0**.

Since **W** is full rank, then only **0** is in N(**W**)

=> **W(S – T) = 0** ⇔ **S – T = 0**

=> **S** = **T** and thus, **S** is unique.

b) From part a, **X = WS** where **S** is unique.

Then **Xb** = **WSb** = **Wc** where **c = Sb**.

Since **S** is unique, so is **c** = **Sb**.

2.19) $\mathbf{U_{.1}} = \mathbf{X_{.1}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, => $D_1 = \sqrt{\mathbf{U'_{.1}U_{.1}}} = 2$ => $\mathbf{Q_{.1}} = D_1^{-1}\mathbf{U_{.1}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

Next, $S_{12} = \frac{\mathbf{U'_{.1}X_{.2}}}{\mathbf{U'_{.1}U_{.1}}} = \frac{10}{4} = \frac{5}{2}$, thus $\mathbf{U_{.2}} = \mathbf{X_{.2}} - S_{12}\mathbf{U_{.1}} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$

$D_2 = \sqrt{\mathbf{U'_{.2}U_{.2}}} = \sqrt{5}$ => $\mathbf{Q_{.2}} = D_2^{-1}\mathbf{U_{.2}} = \begin{bmatrix} -3/2\sqrt{5} \\ -1/2\sqrt{5} \\ 1/2\sqrt{5} \\ 3/2\sqrt{5} \end{bmatrix}$

$S_{13} = \frac{\mathbf{U'_{.1}X_{.3}}}{\mathbf{U'_{.1}U_{.1}}} = 24/4 = 6$, and $S_{23} = \frac{\mathbf{U'_{.2}X_{.3}}}{\mathbf{U'_{.2}U_{.2}}} = 10/5 = 2$,

so $\mathbf{U_{.3}} = \mathbf{X_{.3}} - S_{13}\mathbf{U_{.1}} - S_{23}\mathbf{U_{.2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ => $D_3 = 0$, $\mathbf{U} = \begin{bmatrix} 1 & -3/2 & 0 \\ 1 & -1/2 & 0 \\ 1 & 1/2 & 0 \\ 1 & 3/2 & 0 \end{bmatrix}$, $Q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{S} = \begin{bmatrix} 1 & 5/2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Thus, $\mathbf{R} = \mathbf{DS} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5/2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 12 \\ 0 & \sqrt{5} & 2\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix}$

So X = US = QR = $\begin{bmatrix} 1 & -3/2 & 0 \\ 1 & -1/2 & 0 \\ 1 & 1/2 & 0 \\ 1 & 3/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5/2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/2\sqrt{5} & 0 \\ 1/2 & -1/2\sqrt{5} & 0 \\ 1/2 & 1/2\sqrt{5} & 0 \\ 1/2 & 3/2\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 & 12 \\ 0 & \sqrt{5} & 2\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix}$