STAT 742 Homework #1 Solutions Fall 2008

A.1 Suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are mutually orthogonal vectors that are linearly dependent. Then there exists a constant c such that $\mathbf{x_1} = c\mathbf{x_2}$ by Result A.2. Now,

$$x_1 - cx_2 = 0 \Rightarrow x_1'(x_1 - cx_2) = 0 \Rightarrow x_1'x_1 - cx_1'x_2 = 0$$
$$\Rightarrow x_1'x_1 = 0, \text{ since } x_1 \text{ and } x_2 \text{ are orthogonal, i.e., } x_1'x_2 = 0.$$
$$\text{But } x_1'x_1 = 0 \Rightarrow x_1 = 0. \text{ This is a contradiction, since } x_1 \neq 0.$$

Hence, $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent.

A.12 Let x and y be vectors in the null space, $N[A]$, of A and let c be a scalar.

Then $Ax = 0$ and $Ay = 0$, and

$$A(x+y) = Ax + Ay = 0 + 0 = 0, \textit{ thus } x + y \in N[A],$$
$$\textit{i.e.}, N[A] \text{ is closed under addition.}$$
$$A(cx) = cAx = c(0) = 0, \textit{ thus } cx \in N[A], \text{ i.e., } N[A] \text{ is closed under scalar multiplication.}$$

A.16 If $E = FA$ where F is nonsingular, need to show $N[E] = N[A]$. Now,

$$x \in N[E] \Leftrightarrow Ex = 0 \Leftrightarrow FAx = 0 \Leftrightarrow F^{-1}FAx = 0$$
$$\Leftrightarrow Ax = 0 \Leftrightarrow x \in N[A].$$

Therefore, $N[E] = N[A]$.

A.18 The rank of A is 2, so the column space will consist of 2 vectors. After doing a row-reduction augmentation on $\mathbf{A} = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}]$, you get for the row-reduced matrix $\mathbf{A^*} = [\mathbf{a_1^*}\ \mathbf{a_2^*}\ \mathbf{a_3^*}]$:

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so here, $-5\mathbf{a_1^*} + 3\mathbf{a_2^*} = \mathbf{a_3^*} \Rightarrow -5\mathbf{a_1} + 3\mathbf{a_2} = \mathbf{a_3}$

So the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ form a basis for the column space of $\mathbf{A}$.

A.19 Since the rank of $\mathbf{A}'$ with 5 columns is 2, the null space consists of 3 vectors. First, find the solution to $\mathbf{A'x} = \mathbf{0}$. The row-reduced matrix of $\mathbf{A}'$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the system of equations has $x_1 + x_4 = 0$ and $x_2 + x_3 + x_5 = 0 \Rightarrow x_1 = -x_4$ and $x_2 = -x_3 - x_5$

So $\mathbf{x} = [-x_4, -x_3 - x_5, x_3, x_4, x_5]' = x_3[0, -1, 1, 0, 0]' + x_4[-1, 0, 0, 1, 0]' + x_5[0, -1, 0, 0, 1]'$
$= x_3\mathbf{v_1} + x_4\mathbf{v_2} + x_5\mathbf{v_3}$

Thus, a set of basis vectors for the null space of A' is $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$.

A.32 To show that $\left[(A'A)^{-1}A'\right] = A^{+}$ is the Moore-Penrose generalized inverse of **A**, we must show that it satisfies the four conditions in the Definition A.11 on page 261:

1. $A\left[(A'A)^{-1}A'\right]A = A(A'A)^{-1}A'A = A$
2. $\left[(A'A)^{-1}A'\right]A\left[(A'A)^{-1}A'\right] = (A'A)^{-1}A'A(A'A)^{-1}A' = (A'A)^{-1}A'$
3. $\left[A(A'A)^{-1}A'\right]' = A(A'A)^{-1}A'$
4. $[(A'A)^{-1}A']A = (A'A)^{-1}A'A = I,$ which is symmetric.

4) $\mathbf{PX'X = QX'X} \Leftrightarrow \mathbf{PX'X - QX'X = 0} \Leftrightarrow \mathbf{(P - Q)X'X = 0}$
=> **(P – Q)X’X(P – Q)’ = 0. Let W = X(P – Q)’. Then we have,**
=> **W’W = 0** => **W’ = 0**
⇔ **PX’ – QX’ = 0**
⇔ **PX’ = QX’**

A.36 **P** being idempotent means **PP** = **P**
So **(I – P)(I – P) = I – IP – PI + PP = I – 2P + P = I – P**, thus **I – P** is idempotent.

A.44 Note that $|\mathbf{I}_m + \mathbf{AB}'| = |\mathbf{I}_n||\mathbf{I}_m - (-\mathbf{A})\mathbf{I}_n^{-1}\mathbf{B}')|$ so that result A.18(e) can be applied.
=> $|\mathbf{I}_m + \mathbf{AB}'| = |\mathbf{I}_m||\mathbf{I}_n - \mathbf{B}'\mathbf{I}_m^{-1}(-\mathbf{A})| = |\mathbf{I}_n + \mathbf{B}'\mathbf{A}|$

A.49 Let A be a $m \times n$ real matrix**.** Since $A'A$ and AA' are both non-negative definite matrices, all their eigenvalues are non-negative**.** In order to prove that the non-zero eigenvalues of these matrices are same, we need to show that the non-zero roots of their characteristic polynomials $\left|A'A - \lambda^2 I_n\right|$ and $\left|AA' - \lambda^2 I_m\right|$ are same. However, using the Result A.18 (e), or a modified version of the Problem A.49, we can see that for non-zero λ,

$$\begin{vmatrix} \lambda I_n & A' \\ A & \lambda I_m \end{vmatrix} = |\lambda I_n|\left|\lambda I_m - A(\lambda I_n)^{-1}A'\right| = |\lambda I_m|\left|\lambda I_n - A(\lambda I_m)^{-1}A'\right|.$$

However, using the Result A.18(d),

$$|\lambda I_n|\left|\lambda I_m - A(\lambda I_n)^{-1}A'\right| = \lambda^{n-m}\left|\lambda^2 I_m - AA'\right| \text{ and}$$

$$|\lambda I_m|\left|\lambda I_n - A(\lambda I_m)^{-1}A'\right| = \lambda^{m-n}\left|\lambda^2 I_n - A'A\right|.$$

Thus for non-zero λ, $\left|A'A - \lambda^2 I_n\right| = 0 \Leftrightarrow \left|AA' - \lambda^2 I_m\right| = 0$. Hence the non-zero eigenvalues of $A'A$ are same as non-zero eigenvalues of AA'.

A.50 a) Since $\mathbf{v}^{(j)}$ is an eigenvector of **A’A** with value λ_j, then $\mathbf{A'Av}^{(j)} = \lambda_j \mathbf{v}^{(j)}$
=> $\mathbf{AA'(Av}^{(j)}\mathbf{)} = \lambda_j(\mathbf{Av}^{(j)})$ => $\mathbf{Av}^{(j)}$ is an eigenvector of **AA’** with the same eigenvalue.
b) Using the result from (a), **AV** are the eigenvectors of **A’A**, and so is **U**.
From the result in A.49, the eigenvalues are the same for both **A’A** and **AA’**.

In addition, $\mathbf{U}$ is normalized, but not $\mathbf{A}$V.

Thus, $\mathbf{AV} = \mathbf{U\Sigma}$ where $\mathbf{\Sigma}$ is some diagonal matrix.

c) Suppose $\mathbf{\Sigma}$ is a diagonal matrix with diagonal entry d_i in position (i, i).

$\mathbf{AV} = [\mathbf{Av_1} \ldots \mathbf{Av_r} \ldots \mathbf{Av_n}] = [d_1\mathbf{u_1} \ldots d_r\mathbf{u_r} \ldots d_n\mathbf{u_n}]$ from the result in (b).

Since the $\mathbf{u_i}$'s are normalized, then $d_i = \lambda_i$ for $i = 1,\ldots,r$ and 0 for $d_i = r + 1,\ldots,n$.

Thus, $\mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

d) $\mathbf{U'AV} = \mathbf{U'U\Sigma}$ using the result from (b) and since $\mathbf{U}$ is orthogonal,

$\Rightarrow \mathbf{U'AV} = \mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ from the result in (c)

e) (i) Let $\mathbf{\Sigma_0} = \begin{bmatrix} \boldsymbol{\Lambda}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. $\mathbf{AA^+A} = \mathbf{U\Sigma V'V\Sigma_0 U'U\Sigma' V'} = \mathbf{U\Sigma\Sigma_0\Sigma V'}$

$\mathbf{\Sigma\Sigma_0\Sigma} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{\Sigma}$ (Thus, $\mathbf{\Sigma_0}$ is the g-inverse of $\mathbf{\Sigma}$)

So $\mathbf{AA^+A} = \mathbf{U\Sigma V'} = \mathbf{A}$

(ii) $\mathbf{A^+AA^+} = \mathbf{V\Sigma_0 U'U\Sigma V'V\Sigma_0 U'} = \mathbf{V\Sigma_0\Sigma\Sigma_0 U'} = \mathbf{A^+}$

(iii) $\mathbf{AA^+} = \mathbf{U\Sigma V'V\Sigma_0 U'} = \mathbf{U\Sigma\Sigma_0 U'} = \mathbf{U}\begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U'}$, thus symmetric

(iv) $\mathbf{A^+A} = V\Sigma_0 U'U\Sigma V' = \mathbf{V\Sigma_0\Sigma V'} = \mathbf{V}\begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{V'}$, thus symmetric

Thus, A^+ is the Moore-Penrose generalized inverse for A.

A.53 Suppose $\mathbf{A}$ is an idempotent matrix, and let λ be an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{x}$.

Then $\mathbf{AAx} = \mathbf{A(Ax)} = \mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{Ax} = \lambda^2\mathbf{x}$

Also, $\mathbf{AAx} = \mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \lambda\mathbf{x} = \lambda^2\mathbf{x} \Leftrightarrow \lambda = 0$ or 1

Therefore, all eigenvalues of an idempotent matrix are either 0 or 1.

A.54 rank$(\mathbf{P})$ = # of non-zero eigenvalues and $\mathbf{P}$ being symmetric means trace$(\mathbf{P}) = \Sigma\lambda_i$ where λ_i are the eigenvalues of $\mathbf{P}$.

Since $\mathbf{P}$ is symmetric and idempotent, all eigenvalues are either 0 or 1 from A.53

So rank$(\mathbf{P})$ = trace$(\mathbf{P})$

A.73 **a)** Since V is symmetric, $|V| = \Pi\lambda_i$. Its Spectral decomposition is given by $V = Q\Lambda Q'$ where Q is orthogonal and Λ is a diagonal matrix of eigenvalues of V.

So $\mathbf{I_p} + t\mathbf{V} = \mathbf{I_p} + t\mathbf{Q\Lambda Q'} = \mathbf{QQ'} + t\mathbf{Q\Lambda Q'} = \mathbf{Q}(\mathbf{I_p} + t\mathbf{\Lambda})\mathbf{Q'}$

Thus, the diagonal entries of $\mathbf{I_p} + t\mathbf{\Lambda}$ are $1 + t\lambda_i$, which are eigenvalues of $\mathbf{I_p} + t\mathbf{V}$.

$\mathbf{I_p} + t\mathbf{V}$ is also symmetric, thus, $|\mathbf{I_p} + t\mathbf{V}| = \Pi(1 + t\lambda_i)$

b) $\Pi(1 + t\lambda_i) = 1 + t\Sigma\lambda_i + t^2\sum_{i\neq j}\lambda_i\lambda_j + \ldots + t^p\lambda_1\ldots\lambda_p$

$\Rightarrow \frac{d}{dt} = \Sigma\lambda_i + 2t\Sigma\lambda_i\lambda_j + \ldots + pt^{p-1}\lambda_1\ldots\lambda_p$

at $t = 0$, $\frac{d}{dt} = \Sigma\lambda_i$ = trace$(\mathbf{V})$ since $\mathbf{V}$ is symmetric.

7) Want to show that $\mathbf{(A + CC')\,(A + CC')^{-}\,(A + CC') = (A + CC')}$

Note that $\mathbf{A^{-1}A = AA^{-1} = I}$ since $\mathbf{A}$ is non-singular.

Also note that $\mathbf{(I + C'AC)^{-}}$ is a generalized inverse of $\mathbf{(I + C'AC)}$

$\mathbf{(A + CC')\,(A + CC')^{-}\,(A + CC')}$

{Plugging in the value of $\mathbf{(A + CC')^{-}}$ }

$= \mathbf{(A + CC')(A^{-1} - A^{-1}C(I + C'A^{-1}C)^{-1}C'A^{-1})(A + CC')}$

{Multiplying the middle term out}

$= \mathbf{(A + CC')A^{-1}(A + CC') - (A + CC')A^{-1}C(I + C'A^{-1}C)^{-}C'A^{-1}(A + CC')}$

{Multiplying the terms between the parentheses}

$\mathbf{= (A + CC')(I + A^{-1}CC') - (C + CC'A^{-1}C)(I + C'A^{-1}C)^{-}(C' + C'A^{-1}CC')}$

{Distributing out $\mathbf{(I + A^{-1}CC')}$ and factoring out the $\mathbf{C}$ and $\mathbf{C'}$ terms in the second term}

$\mathbf{= (A + CC') + (A + CC')(A^{-1}CC') - C(I + C'A^{-1}C)(I + C'A^{-1}C)^{-}(I + C'A^{-1}C)C'}$

{Multiplying out second term and using the fact that $\mathbf{(I + C'AC)^{-}}$ is a g-inverse}

$\mathbf{= (A + CC') + (CC' + CC'A^{-1}CC') - C(I + C'A^{-1}C)C'}$

{Factoring out the $\mathbf{C}$ and $\mathbf{C'}$ from the second term}

$\mathbf{= (A + CC') + C(I + C'A^{-1}C)C' - C(I + C'A^{-1}C)C'}$

$\mathbf{= (A + CC')}$

Thus, the generalized inverse of $\mathbf{(A + CC')}$ is $\mathbf{A^{-1} - A^{-1}C(I + C'A^{-1}C)^{-}C'A^{-1}}$.