

****READ ME****

- This take home portion of the Final Examination consists of five problems, for a total of 160 points (40% of the Overall Course Score). Individual scores for each problem are given on the left margin.
- You will need your solutions to answer the questions on the In-class portion of the Final Examination scheduled at 11:30AM on Wednesday, December 10, 2008. Both parts will be due at the end of the In-Class Exam.
- Please remember that this is an *Examination*. It is on your honor to work on these problems *by yourself*. No assistance from, or discussion/collaboration with, any other person or a website is permitted. If you violate this rule, your answers on the in-class portion will definitely provide indications that you may not have worked on the Take-Home portion on your own.
- Even a slight indication of violation of this rule will constitute a *serious breach of trust*, and lead to disciplinary action.
- Of course, if you do not make an effort to learn to solve these problems now, QII will become the time to worry.



- [30 points] Consider the general linear model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\mathbf{Y}'_n = (Y_1, Y_2, \dots, Y_n)$, $\mathbf{X}' = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \end{pmatrix}$ is a $2 \times n$ matrix, $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, and $\boldsymbol{\varepsilon}' = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is random error, with $E[\boldsymbol{\varepsilon}] = \mathbf{0}$, $\text{var}[\boldsymbol{\varepsilon}] = \sigma^2 \mathbf{V}$. Assume that $\mathbf{V} = \text{diag}(v_{ii}, i = 1, 2, \dots, n)$, a diagonal matrix with $v_{ii} = x_{ii}^2$; and that the design matrix \mathbf{X} is of full rank.
 - [8 points] Obtain the generalized least squares (GLS) estimator, $\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$. [You must provide the explicit expression to receive credit.]
 - [8 points] Derive the variance-covariance matrix $\text{Cov}(\hat{\boldsymbol{\beta}})$ of $\hat{\boldsymbol{\beta}}$. Provide conditions, if any, on \mathbf{X} , under which $\hat{\beta}_1$ and $\hat{\beta}_2$ are uncorrelated.
 - [8 points] Given that $\varepsilon_i, i = 1, 2, \dots, n$ are normally distributed, find a 90% simultaneous confidence region for the parameters β_1, β_2 , as well as Bonferroni confidence intervals for each of these parameters.
 - [6 points] Given that $\varepsilon_i, i = 1, 2, \dots, n$ are normally distributed, describe the critical region for the α -level t-test for testing the hypotheses $H_0 : \beta_1 = \beta_2$ vs. $H_1 : \beta_1 > \beta_2$.

2. [40 points] Consider a two-factor experiment consisting of 8 observations, with factor A at two levels, factor B at two levels, and two observations per cell, i.e.,

$$\Omega: \left\{ \begin{array}{l} Y_{ijk} = \mu + \alpha_i + \tau_j + \gamma_{ij} + \varepsilon_{ijk}, \\ i, j, k = 1, 2; \\ E[\varepsilon_{ijk}] = 0, \text{Var}(\varepsilon_{ijk}) = \sigma^2; \\ \varepsilon_{ij} \text{ 's are normally distributed,} \\ \text{Corr}(\varepsilon_{i1k}, \varepsilon_{i2k}) = \rho, \text{ for each } (i,k), \\ \text{and all other pairs of } \varepsilon_{ijk} \text{ are uncorrelated.} \end{array} \right.$$

- a. [7 points] Write this model in the general linear model notation, i.e., define the parameter vector, the matrix \mathbf{X} , the error vector ε , and its covariance matrix in the form $\sigma^2 V$. What is the rank of the matrix \mathbf{X} in this model?
- b. [4 points] Write the general linear model for the cell means

$$\bar{Y}_{ij.} = \frac{1}{2} \sum_{k=1}^2 Y_{ijk}, i, j = 1, 2.$$

Find the rank of the matrix \mathbf{X} for this model.

- c. [7 points] Show that the distribution of $(Y_{ijk} - \bar{Y}_{ij.})$ involves only the parameter σ^2 . Find the distribution of the SSE in the full model.
- d. [7 points] Derive the necessary and sufficient conditions for the linear function

$$\sum_{i=1}^2 \sum_{j=1}^2 c_{ij} Y_{ij}$$

to be estimable. Explain how many linearly independent functions of interactions are estimable. Describe the F-test for testing the hypothesis that these linear function(s) are equal to zero.

- e. [8 points] Is the OLS estimator for $(\tau_1 - \tau_2)$ in this model equal to its best linear unbiased estimator? Explain.
- f. [7 points] Find the F-tests for testing the hypothesis

$$H_0: \tau_1 = \tau_2 \text{ against } H_1: \tau_1 \neq \tau_2,$$

in the full model, as well as in the model without interactions.

3. [20 points] Consider the general one-way ANOVA model in Chapter 3, with n_i observations in the i^{th} group, $i=1, \dots, a$, and $N = \sum_i n_i$.

a. [10 points] Write the usual non-estimable constraint $\sum n_i \alpha_i = 0$ in a vector notation $c' \beta$ for the parameter vector $\beta = (\mu, \alpha_1, \dots, \alpha_a)$. Show that

$$(X'X + cc')^{-1} = \begin{bmatrix} \frac{N+1}{N^2} & -\frac{1}{N^2} \mathbf{1}'_a \\ -\frac{1}{N^2} \mathbf{1}_a & D^{-1} - \frac{N-1}{N^2} \mathbf{1}_a \mathbf{1}'_a \end{bmatrix},$$

where, $D = \text{diag}(n_1, \dots, n_a)$. Find the solution to the normal equations using this result, and show that the solution satisfies the constraint. If you use any result from matrix algebra, please give an appropriate reference from the book or class handouts.

b. [10 points] In the normal linear model, i.e., $Y \sim \text{Normal}_N(X\beta, \sigma^2 I_N)$, assume that the column rank of X is full. Find the conditional distribution of $l'Y$ given $X'Y$.

4. [40 points] Text Book Problem # 7.11 (a, b, c, e, h), with 8 points for each part.

5. [30 points] Let $Y_N \sim N(\mu \mathbf{1}_N, \sigma^2 V)$ be a N -dimensional observation vector, with parameters (μ, σ^2) .

a. [10 points] Suppose that $N=3$, and $V = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}$. Find the joint distribution of

$(\ell'_1 Y, \ell'_2 Y)$, where $\ell'_1 = (1, 1, 1)$ and $\ell'_2 = (1, -1, -1)$. Note that the vectors (ℓ_1, ℓ_2) are not orthogonal. Find the value(s) of ρ for which $\ell'_1 Y$ and $\ell'_2 Y$ are independently distributed?

b. [10 points] Suppose that $N=4$, $\mu=0$, and $V=I$. Let $Q = Y_1 Y_2 + Y_3 Y_4$. Prove that the random variable Q/σ^2 does not have a χ^2 distribution.

c. [10 points] Let the matrix V be non-singular, and the matrix A be idempotent. Prove that $Y'AY$ is distributed as a linear combination of independent non-central χ^2 random variables. State the conditions on A and V , under which $Y'AY \sim \sigma^2 \chi^2$.