Fall 2007 **Final Examination** **STAT 742**
(*Take Home Portion*) **Dr. Goel**

****READ ME****

a. This take home portion of the Final Examination consists of five problems, for a total of 140 points. Individual scores for each problem are given on the left margin.
b. You will need your solutions to answer the questions on the In-class portion of the Final Examination scheduled at 1:30PM on Tuesday, December 4, 2007. Both parts will be due at the end of the In-Class Exam.
c. This is an *Examination*. It is on your honor to work on these problems *by yourself*. Assistance from, or collaboration with, any other individual is not permitted. If you violate this rule, your answers on the in-class portion will give some indications.
d. Even a slight indication of violation of this rule will constitute a *serious breach of trust*, and may lead to disciplinary action.

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1. [30 points] Matrices and Quadratic Forms:
   a. [7 points] Let $Y \sim N(\mu, V)$, where V is a possibly singular matrix. Let A be a symmetric non-negative definite matrix. Find the sampling distribution of $Y'AY$. [Hint: In general, the distribution of $Y'AY$ is a linear combination of independent non-central χ^2 random variables.]

   b. [8 points] Show that the distribution of $Y'AY$ is a non-central chi-square if and only if $VAVAV = VAV$, and its degrees of freedom equals Rank(AV).

   c. [15 points] For a one factor random effect model, use the result in part (a) above to show that SSA and SSE have chi-square distributions with relevant degrees of freedom.

2. [20 points] For the general linear model $Y = X\beta + \varepsilon$, with uncorrelated errors having mean zero and variance σ^2, suppose that the design matrix $\mathbf{X} = [X_1 \mid X_2]$, and $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$.
   a. [10 points] Show that every solution of $\mathbf{X}_1'\mathbf{X}_1\tilde{\beta}_1 = \mathbf{X}_1'\mathbf{Y}$, [i.e., every OLS estimator of β_1 under $\mathrm{H}_0 : \beta_2 = 0$] satisfies $\mathbf{X}'\mathbf{X}\hat{\beta} = \mathbf{X}'\mathbf{Y}$ if and only if $\mathbf{X}_1'\mathbf{X}_2\hat{\beta}_2 = 0$, [*i.e.*, $\mathbf{X}_2\hat{\beta}_2 \perp \Re[\mathbf{X}_1]$ for all $\hat{\beta}_2$].

   b. [10 points] Does $\mathbf{X}_2\hat{\beta}_2 \perp \Re[\mathbf{X}_1]$ for all $\hat{\beta}_2$ imply that $\Re[\mathbf{X}_2] \perp \Re[\mathbf{X}_1]$? Explain.

3. [30 points] Suppose that $X_{n_1\times 1}, Y_{n_2\times 1}, Z_{n_3\times 1}$ are mutually independent random vectors, such that elements of each vector are uncorrelated random variables each having same variance σ^2. However, their means are given by $E[X_{n_1\times 1}] = \theta_1\mathbf{1}_{n_1\times 1}, E[Y_{n_2\times 1}] = \theta_2\mathbf{1}_{n_2\times 1}, E[Z_{n_3\times 1}] = (\theta_1 - \theta_2)\mathbf{1}_{n_3\times 1}$. Find the Best Linear Unbiased Estimators of (θ_1, θ_2) and their variance-covariance matrix.

4. [30 points] Consider a two-way classification model with factor A at 2 levels and factor B at J levels, and one observation per cell, i.e.,

$$\Omega : \begin{cases} Y_{ij} = \mu + \alpha_i + \tau_j + \varepsilon_{ij}, \\ i = 1, 2;\ j = 1, 2, \cdots, J; \\ E[\varepsilon_{ij}] = 0, Var(\varepsilon_{1j}) = \sigma^2, Var(\varepsilon_{2j}) = 2\sigma^2; \\ \varepsilon_{ij}\text{'s are normally distributed,} \\ Corr(\varepsilon_{1j}, \varepsilon_{2j}) = \rho,\ j = 1, 2, \cdots, J, \\ \text{all other } \varepsilon_{ij}\text{'s are uncorrelated.} \end{cases}$$

a. [7 points] Write the model in the general linear model notation, i.e., define the parameter vector, the matrix **X,** the error vector ε, and its covariance matrix in the form $\sigma^2 V$.

b. [8 points] Derive a necessary and sufficient condition for

$$\sum_{i=1}^{2} c_i \alpha_i + \sum_{j=1}^{J} d_j \tau_j$$

to be estimable.

c. [7 points] Is the OLS estimator for $(\alpha_1 - \alpha_2)$ in this model equal to its Gauss-Markov estimator? Explain your answer and obtain the Gauss-Markov estimator of $(\alpha_1 - \alpha_2)$.

d. [8 points] Show that the F-test for $H_0 : \alpha_1 = \alpha_2$ against $H_1 : \alpha_1 \neq \alpha_2$, and the two-sided t-test for testing $H_0 : E[d_j] = 0$ against $H_1 : E[d_j] \neq 0$ based on the pair-wise differences $d_j = Y_{1j} - Y_{2j},\ \ j = 1, 2, \cdots J$, are equivalent.

5. [30 points] In the set up of the Exercise 8.5, assume that the true regression lines are parallel with equal slope β_1. Let η be the true horizontal distance between the two regression lines, (i.e., for a fixed y coordinate, η = distance between the corresponding inverse image of y on each line).

a) [10 points] For the statistic $d = \hat{\beta}_{1,0,H} - \hat{\beta}_{2,0,H} + \hat{\beta}_{1,H}\eta$, find the mean and variance, $\sigma^2 C$.

b) [10 points] Find the estimate $\hat{\sigma}^2$ of σ^2 for this model, and show that $\hat{\sigma}^2$ and d are independently distributed.

c) [10 points] Show that the confidence interval for η is given by solving the quadratic function of η, namely, $\dfrac{d^2}{\hat{\sigma}^2 C} = F_{1, n_1 + n_2 - 3, \alpha}$.