

## STAT 893: Problem Set #2

### Problem 1: (2.14)

From Problem 2.7 (on the last problem set), we know that the eigenvalues of  $\mathbf{x}\mathbf{x}'$  are all zero except for one, which is equal to  $\mathbf{x}'\mathbf{x}$ .

Hence,

a) Yes,  $\frac{\mathbf{x}\mathbf{x}'}{\mathbf{x}'\mathbf{x}} = \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)'$  is symmetric and idempotent because its eigenvalues are 0 and 1.

b) No, in general we cannot claim  $\mathbf{x}\mathbf{x}'$  to be idempotent because the eigenvalues are zero and  $\mathbf{x}'\mathbf{x}$ , which is not in general either 0 or 1.

### Problem 2: (2.22)

$$\begin{aligned} |\mathbf{P}'\mathbf{A}\mathbf{P} - \lambda\mathbf{I}| &= |\mathbf{P}'\mathbf{A}\mathbf{P} - \lambda\mathbf{P}'\mathbf{P}| \\ &= |\mathbf{P}'\mathbf{A}\mathbf{P} - \mathbf{P}'\lambda\mathbf{I}\mathbf{P}| \\ &= |\mathbf{P}'(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}| \\ &= |\mathbf{P}'||\mathbf{A} - \lambda\mathbf{I}||\mathbf{P}| \\ &= |\mathbf{A} - \lambda\mathbf{I}||\mathbf{P}'||\mathbf{P}| \\ &= |\mathbf{A} - \lambda\mathbf{I}| \end{aligned}$$

So  $\mathbf{A}$  and  $\mathbf{P}'\mathbf{A}\mathbf{P}$  have the same eigenvalues. Under our hypothesis, then, they are both idempotent. Symmetric is straightforward:

$$\begin{aligned} (\mathbf{P}'\mathbf{A}\mathbf{P})' &= \mathbf{P}'\mathbf{A}'(\mathbf{P}')' \\ &= \mathbf{P}'\mathbf{A}'\mathbf{P} \\ &= \mathbf{P}'\mathbf{A}\mathbf{P} \end{aligned}$$

because  $\mathbf{A}$  is symmetric.

### Problem 3: (2.23)

The eigenvalues of  $\mathbf{A}$  are 4.866, 1.789, and 0.345, all of which are positive, so  $\mathbf{A}$  is p.d.

Also, the step-wise technique we discussed for decomposing matrices gives

$$\mathbf{B} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{3} \end{bmatrix} \text{ satisfies } \mathbf{A} = \mathbf{B}\mathbf{B}'.$$

### Problem 4: (2.24)

Note that  $\mathbf{J}_n = \mathbf{1}\mathbf{1}'$ , so its eigenvalues are zero (with multiplicity  $n-1$ ) and  $n$ , again by problem 2.7. Hence,  $\mathbf{J}_n$  is p.s.d. We know that the identity matrix is p.d. (either from the eigenvalues, or looking at the quadratic form polynomial, as we did in class). Then, so long as  $a \in (0, 1)$ ,  $(1-a)\mathbf{I}_n$  is p.d. and  $a\mathbf{J}_n$  is p.s.d., so their sum is p.d. In the other two cases, we have either just  $\mathbf{I}_n$ , which is p.d., or just  $\mathbf{J}_n$ , which is not. Hence,  $(1-a)\mathbf{I}_n + a\mathbf{J}_n$  is p.s.d. for all  $a \in [0, 1)$ .

### Problem 5: (2.26)

a) Letting  $\mathbf{x}' = [x_1 \ x_2 \ \cdots \ x_n]$ ,

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_ix_j$$

and as a polynomial, it can only be identically zero (that is, zero for all values of  $x_i$ ) if it is the zero polynomial.

b) Because  $\mathbf{A}$  is chosen to be symmetric, it has  $n$  LIN eigenvectors  $\mathbf{v}_i$ , all non-negative. Specifically, we can write

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

So,

$$\begin{aligned}\mathbf{x}'\mathbf{A}\mathbf{x} &= \mathbf{x}'\mathbf{A}(c_i\mathbf{v}_i) \\ &= \mathbf{x}' \sum c_i \lambda_i \mathbf{v}_i \\ &= \sum c_i^2 \lambda_i \mathbf{v}_i' \mathbf{v}_i \\ &= \sum \lambda_i c_i \|\mathbf{v}_i\|\end{aligned}$$

which is non-negative, and only zero if the only nonzero  $c_i$  come for the zero eigenvalues; that is, if  $\mathbf{x}$  is in the eigenspace associated with  $\lambda = 0$ . But that would imply that  $\mathbf{A}\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ .