STAT 893: Problem Set #2

Problem 1: (2.3)

Calculating the determinant in both ways gives us

$$\begin{aligned}|\mathbf{A}| &= |\mathbf{P}| \cdot |1 - \mathbf{x}'\mathbf{P}^{-1}\mathbf{x}| \\ &= 1 \cdot |\mathbf{P} - \mathbf{x}1\mathbf{x}'|\end{aligned}$$

Now, notice that $\mathbf{x}'\mathbf{P}^{-1}\mathbf{x}$ is a $1{\times}1$ matrix, so we can treat is as just a scalar and drop the determinants from that term. Hence,

$$\begin{aligned}&|\mathbf{P}||1 - \mathbf{x}'\mathbf{P}^{-1}\mathbf{x}| = 1 \cdot |\mathbf{P} - \mathbf{x}1\mathbf{x}'| \\ \Leftrightarrow &1 - \mathbf{x}'\mathbf{P}^{-1}\mathbf{x} = |\mathbf{P} - \mathbf{x}\mathbf{x}'|/|\mathbf{P}| \\ \Leftrightarrow &\mathbf{x}'\mathbf{P}^{-1}\mathbf{x} = 1 - |\mathbf{P} - \mathbf{x}\mathbf{x}'|/|\mathbf{P}|\end{aligned}$$

Problem 2: (2.4) a) Certainly, the characterization of rank as the number of linearly independent column vectors makes the first equality obvious, but also note that

$$\begin{bmatrix}\mathbf{0} & \mathbf{B} \\ \mathbf{A} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}\end{bmatrix}$$

And since the first matrix is non-singular, it does not affect ranks.

b) Here, we note that

$$\begin{bmatrix}\mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{AB} \\ \mathbf{CA} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{D} & \mathbf{CA} \\ \mathbf{AB} & \mathbf{A}\end{bmatrix}$$

Now, we also note that

$$\begin{bmatrix}\mathbf{I} & -\mathbf{C} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{D} & \mathbf{CA} \\ \mathbf{AB} & \mathbf{A}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ -\mathbf{B} & \mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{D} - \mathbf{CAB} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}\end{bmatrix}$$

and we can apply part (a) to finish the problem, because each of the triangular matrices are nonsingular, so do not affect ranks. (Working in two steps here, to eliminate first one off-diagonal block then the other works, too.)

Problem 3: (2.6)

a) Again calculating the determinant two ways, we get

$$\begin{aligned}\begin{vmatrix}-\lambda\mathbf{I}_n & -\mathbf{A} \\ \mathbf{B} & \mathbf{I}_k\end{vmatrix} &= |\mathbf{I}_k|| - \lambda\mathbf{I}_n - (-\mathbf{A})\mathbf{I}_k^{-1}\mathbf{B}| \\ &= |\mathbf{AB} - \lambda\mathbf{I}_n| \\ \text{and also,}\quad &= | - \lambda\mathbf{I}_n||\mathbf{I}_k - \mathbf{B}(-\lambda\mathbf{I}_n)^{-1}(-\mathbf{A})| \\ &= (-\lambda)^n|\mathbf{I}_k - \frac{1}{\lambda}\mathbf{BA}| \\ &= (-\lambda)^{n-k}| - \lambda\mathbf{I}_k + \mathbf{BA}| \\ &= (-\lambda)^{n-k}|\mathbf{BA} - \lambda\mathbf{I}_k|\end{aligned}$$

Note that because $\mathbf{BA}$ is $k \times k$ the powers on λ work out.

b) The eignevalues of $\mathbf{AB}$ are the solutions of the equation

$$|\mathbf{AB} - \lambda\mathbf{I}_n| = (-\lambda)^{n-k}|\mathbf{BA} - \lambda\mathbf{I}_k| = 0$$

which must be 0 with multiplicity $(n-k)$ and whatever the k eigenvalues of $\mathbf{BA}$ are.

c) Let $\mathbf{v}$ be a nonzero eigenvalue of $\mathbf{AB}$. Then

$$\begin{aligned} &\mathbf{ABv} = \lambda\mathbf{v} \\ \Rightarrow\ &\mathbf{B}(\mathbf{AB})\mathbf{v} = \mathbf{B}(\lambda\mathbf{v}) \\ \Rightarrow\ &(\mathbf{BA})\mathbf{Bv} = \lambda(\mathbf{Bv}) \\ \Rightarrow\ &\mathbf{BAu} = \lambda\mathbf{u} \end{aligned}$$

so $\mathbf{u}$ is an eigenvector of $\mathbf{BA}$, with eigenvalue λ.

Problem 4: (2.7)

Because $r(\mathbf{aa}') \leq \min\{r(\mathbf{a}), r(\mathbf{a}')\} = 1$, and the only matrix with rank 0 is the zero matrix (not an option here – $\mathbf{a}$ was assumed to be nonzero), we know that $\mathbf{A}$ has one nonzero eigenvalue. (Ok, to be fair, I'm using problem 12 here. Perfectly legal – we know $\mathbf{A}$ is symmetric, but go read problem 12 now if it makes you feel better.)

But we have already established that $tr(\mathbf{aa}') = tr(\mathbf{a}'\mathbf{a})$, and the trace is just the sum of the eigenvalues. Since we only have one eigenvalue in each case, they must be equal.

Problem 5: (2.12)

Because $\mathbf{A}$ is symmetric, we know we can form a matrix $\mathbf{Q}$ whose columns are orthonormal eigenvectors, and specifically, $\mathbf{Q}'\mathbf{AQ} = \mathbf{D}$ where $\mathbf{Q}$ is nonsingular and $\mathbf{D}$ is diagonal.

But then the number of linearly independent columns of $\mathbf{D}$ is exactly the number of nonzero columns, which is itself exactly the number of nonzero eigenvalues (counting repetitions). But since $\mathbf{Q}$ was nonsingular, $r(\mathbf{A}) = r(\mathbf{Q}'\mathbf{AQ}) = r(\mathbf{D})$.