STAT 693: Problem Set #1

Problem 1: Find a basis for the vector space defined by

$$\{\mathbf{v} = (x, y, z, w) \in \Re^4 : 3x - y - z + w = 0\}$$

and determine its dimension.

Solution: Our goal is to find as many linearly independent vectors as possible that satisfy the condition $3x - y - z + w = 0$. This does not need to be too complex; we can choose values for the first three components as we desire, then find the correct value of w to make the vector an element in our space (btw, this leads to a good understanding of degrees of freedom and dimension).

Starting with $x = 1,\ y = 0,\ z = 0$ gives us the vector $\mathbf{v}_1 = [1\ \ 0\ \ 0\ \ -3]'$. Does this span the space? No, we certainly have vectors with non-zero values for y and z. So, we can similarly construct $mathbf{v}_2 = [0\ \ 1\ \ 0\ \ 1]'$ and $\mathbf{v}_3 = [0\ \ 0\ \ 1\ \ 1]'$.

How are we sure this is a basis? The three vectors are obviously (NOTE: always be careful whenever anyone uses the word "obviously", and check that you really can verify the claim being made) linearly independent. Plus, this space must be smaller than $\mathcal{R}$ (why?), so it must have dimension less than 4. We must have a spanning set, then, so we're done.

Problem 2: In $\Re^2$, write the vector $\mathbf{x} = [2\ \ -1]'$ in terms of the basis vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Solution: Note that $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis, so we know a linear combination can be found.

$$\begin{aligned} \begin{bmatrix} 2 \\ -1 \end{bmatrix} &= a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &\Leftrightarrow \begin{cases} 2 = a_1 - a_2 \\ -1 = 2a_1 + 2a_2 \end{cases} \\ &\Leftrightarrow a_1 = 3/4,\ a_2 = -5/4 \end{aligned}$$

Problem 3: Let H be the vector space defined as

$$H = \{(x, y, z) \in \Re^3 | x - y - z = 0\}$$

and let $\mathbf{v} = [-1\ 2\ 4]'$. Then do the following:

- Find an orthonormal basis for H.
- Find $\text{proj}_H \mathbf{v}$.
- Find an orthonormal basis for $H^\perp$.
- Write $\mathbf{v}$ as $\mathbf{h} + \mathbf{p}$. where $\mathbf{h} \in H$ and $\mathbf{p} \in H^\perp$.

Solution: There are many choices for a basis; each one will give its own orthonormal basis. One choice of basis is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

which gives $\mathbf{u}_1 = \frac{1}{\sqrt{2}}[1 \;\; 0 \;\; 1]'$ and

$$\begin{aligned} \mathbf{v}_2' &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \left(-\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \\ \Rightarrow \mathbf{u}_2 &= \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}. \end{aligned}$$

Therefore the projection of $\mathbf{v}$ onto H is

$$\frac{3}{\sqrt{2}}\mathbf{u}_1 - \frac{1}{\sqrt{6}}\mathbf{u}_2 = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

Note that if you multiply this out, the vector you end up with is the same *regardless of what basis you started with*, because the projection is only dependent on H.

As far as $H^\perp$ is concerned, note that not every vector not in H is in $H^\perp$ – for example, $\mathbf{v}$ is in neither space. But, $\mathbf{v} - \text{proj}_H \mathbf{v}$ is in $H^\perp$, and that vector is equal to $\begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$. Normalizing this gives $\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, an orthonormal vector for $H^\perp$, since it is only a one-dimensional space.

Finally, we note that $h = \text{proj}_H \mathbf{v}$, and $p = \mathbf{v} - \text{proj}_H \mathbf{v}$, each of which have already been calculated.

<u>Problem 4:</u> Determine whether the following vectors constitute a linearly independent set.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Solution: We attempt to find a non-zero solution to the system

$$\begin{aligned} 1a_1 + 2a_2 - 3a_3 &= 0 \\ 2a_2 + 0a_2 + 1a_3 &= 0 \\ 3a_1 + 1a_2 + 1a_3 &= 0 \\ 4a_1 - 1a_2 + 2a_3 &= 0 \end{aligned}$$

Solving the first three equations gives the unique solution $a_1 = a_2 = a_3 = 0$, which is clearly a solution to the last equation as well. Note: the zero solution is always going to work for this setup; linear dependence requires *more* than one solution to the system.