Solutions

Stat 428  Midterm I, Spring, 2005

There are five problems (100 points in total) in the exam. You can use one sheet of paper and calculator in the exam but you may not share the sheet and calculator. Please SHOW your work. (Explain and/or show formulas. Numbers with no hint of how you got them will get NO credit.) Please write your answers on the pages with the problems. GOOD LUCK.

1. Let \( X_1, X_2, \ldots, X_5 \), be independent Normal random variables with

\[
E(X_1) = 2\mu, \quad E(X_2) = \mu, \quad E(X_3) = 3\mu, \quad E(X_4) = 2\mu, \quad E(X_5) = \mu. \]

All of the \( X \)'s have Variance = \( \mu \).

Let

\[
Y = \left( \frac{X_1 + X_2}{2} \right) - \left( \frac{X_3 + X_4 + X_5}{3} \right)
\]

a) (5 points) Find \( E(Y) \).

\[
E(Y) = \left( \frac{E(X_1) + E(X_2)}{2} \right) - \left( \frac{E(X_3) + E(X_4) + E(X_5)}{3} \right) = \frac{2\mu + \mu}{2} - \frac{3\mu + 2\mu + \mu}{3} = \frac{3\mu}{2} - \frac{6\mu}{3} = 1.5\mu - 2\mu = -0.5\mu
\]

b) (5 points) Use the result of a) to find a value of \( c \) so that \( cY \) is an unbiased estimator of \( \mu \). Be sure to justify that your estimator is unbiased.

Let \( c = -2 \). Use: \( -2Y \).

Justify: \( E(-2Y) = -2E(Y) = -2(-0.5\mu) = \mu \)

ALTERNATIVELY: \( E(cY) = cE(Y) = c(-0.5\mu) = \mu \Rightarrow -0.5c = 1 \Rightarrow c = -2 \)

(10 points) Suppose that \( \mu = 1 \). Find \( P(Y \geq 3) \).

\( Y \) is a linear combination of Normals, so \( Y \) is Normal. \( E(Y) = -0.5 \), and

\[
Var(Y) = \frac{Var(X_1) + Var(X_2)}{4} + \frac{Var(X_3) + Var(X_4) + Var(X_5)}{9}
\]

\[
= \frac{12 + 18}{4} + \frac{9}{9} = \frac{3}{2} + 1 = 3 + 2 = 5
\]

\[
P(Y \geq 3) = P\left(Z \geq \frac{3 - (-0.5)}{ \sqrt{5} } \right) = P\left(Z \geq \frac{3.5}{ \sqrt{5} } \right) = P\left(Z \geq \frac{3.5}{2.236} \right) = P(Z \geq 1.57)
\]

\[
= 1 - P(Z \leq 1.57) = 1 - 0.9418 = 0.0582
\]
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2. (10 points) When \( X \) has density
\[
f(x) = 0.5(1 + \theta x) \quad -1 \leq x < 1
\]
\[= 0 \quad \text{otherwise},
\]
it can be shown that \( E(X) = \frac{\theta}{3} \). Use this fact to find the moment method estimator of \( \theta \) based on a random sample of size \( n \) from this distribution.

The moment method says to equate the sample average, \( \bar{X} \), to the expected value and solve for \( \theta \). Call the solution \( \hat{\theta} \). Thus, we get:
\[
\bar{X} = \frac{\hat{\theta}}{3} \Rightarrow \hat{\theta} = 3 \bar{X}.
\]

3. (10 points) It was reported that, in a random sample of 507 households in a certain area, 142 owned firearms.
Calculate a (two-sided) confidence interval using a 98% confidence level for the proportion of all households in this area that own firearms. Use the simplified formula.

98% Confidence \( \Rightarrow \alpha = 0.02 \Rightarrow \frac{\alpha}{2} = 0.01 \). \( z_{0.01} = 2.326 \)

\[
\hat{p} = \frac{X}{n} = \frac{142}{507} = 0.28
\]

\( CI \) is:
\[
\hat{p} \pm z_{0.01} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}
\]
\[
= 0.28 \pm 2.326 \sqrt{\frac{(0.28)(0.72)}{507}} = 0.28 \pm 2.326 \sqrt{0.0004} = 0.28 \pm 2.326(0.0199)
\]
\[
= 0.28 \pm 0.0464 = (0.23, 0.33)
\]
4. Resonance frequency measurements were made on a random sample of 25 tennis rackets. It is assumed that the distribution of resonance frequencies is Normal with \( \sigma = 2 \). A confidence interval for the true mean resonance frequency, \( \mu \), is calculated to be \((114.17, 116.03)\).

   a) (7 points) Find the value of the sample mean, \( \bar{X} \).

\[
\bar{X} \text{ is in the middle of the interval: } \\
\bar{X} = \frac{114.17 + 116.03}{2} = 115.1
\]

b) (13 points) Find the confidence level that was used to compute the interval above.

The width of the interval is:

\[
\frac{2z\sigma}{\sqrt{n}} = \frac{2(2)}{\sqrt{25}} = \frac{4z}{5} = 0.8z = 116.03 - 114.17 = 1.86
\]

\[\Rightarrow z = \frac{1.86}{0.8} = 2.325\]

From the tables, \(2.325 \geq z_{0.01} = z_{\frac{\alpha}{2}}\). Thus, \(\frac{\alpha}{2} = 0.01\) and \(\alpha = 0.02 = 2\%\). This is a 98\% CI.

c) (15 points) If the engineers want an interval width of 0.5 for a confidence level of 95\%, what sample size would they need?

For a 95\% CI, \(z = 1.96\).

\[
w = \frac{2z(2)}{\sqrt{n}} \Rightarrow \sqrt{n} = \frac{4z}{w} = \frac{4(1.96)}{0.5} = 8(1.96) = 15.68
\]

\[n = 245.86 \text{ Use } n = 246.\]
5. A random sample of \( n = 9 \) observations on stabilized viscosity of asphalt gave a sample mean of 2884.3 and sample standard deviation of 81.6. Assume that the population distribution of stabilized viscosities is Normal.

a) (15 points) Calculate a 90% upper confidence bound for the true average stabilized viscosity, \( \mu \).

\[ \text{df} = n-1 = 8. \text{ For a 90\% bound, we need} \]
\[ t_{0.1,8} = 1.397 \]

A 90% Upper conf. bound is 
\[ \bar{x} + (t_{0.1,8}) \frac{s}{\sqrt{n}} = 2884.3 + \frac{(1.397)(81.6)}{\sqrt{9}} \]
\[ = 2884.3 + \frac{113.995}{3} = 2884.3 + 37.998 = 2922.3 \]

b) (10 points) Calculate a 90% upper prediction bound for one future observation on the stabilized viscosity of asphalt.

\[ t_{0.1,8} = 1.397 \]

A 90% Upper prediction bound is 
\[ \bar{x} + (t_{0.1,8})s\sqrt{1 + \frac{1}{n}} = 2884.3 + (1.397)(81.6)\sqrt{1 + \frac{1}{9}} \]
\[ = 2884.3 + 113.995\sqrt{\frac{10}{9}} = 2884.3 + \frac{113.995\sqrt{10}}{3} = 2884.3 + 37.998(3.16) = 2884.3 + 120.16 = 3004.5 \]