Chapter 5

*38.a.

<table>
<thead>
<tr>
<th>X_1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>p(x_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X_2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>p(x_1)</td>
<td>0.2</td>
<td>0.5</td>
<td>0.3</td>
<td></td>
</tr>
</tbody>
</table>

The cells in the table above show the value of the sum, T_0. The probability of any cell is the product of the marginal probabilities. The probability m.f. for T_0 is found by adding the probabilities for the cells for which T_0 has the given value.

Thus, P(T_0 = 0) = p(0,0) = (.2)(.2) = 0.04, P(T_0 = 1) = p(1,0) + p(0,1)= (.5)(.2) + (.2)(.5) = 0.2, etc.

The table below summarizes the pmf.

<table>
<thead>
<tr>
<th>T_0</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(T_0)</td>
<td>.04</td>
<td>.20</td>
<td>.37</td>
<td>.30</td>
<td>.09</td>
</tr>
</tbody>
</table>

b. \( \mu_{T_0} = E(T_0) = 2.2 = 2 \cdot \mu \)

c. \( \sigma^2_{T_0} = E(T_0^2) - (E(T_0))^2 = 5.82 - (2.2)^2 = .98 = 2 \cdot \sigma^2 \)

*46 \( \mu = 12 \text{ cm} \quad \sigma = .04 \text{ cm} \)

a. \( E(\bar{X}) = \mu = 12 \text{ cm} \)

\[ \sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{.04}{\sqrt{4}} = .01 \text{ cm} \]

b. \( E(\bar{X}) = \mu = 12 \text{ cm} \)

\[ \sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{.04}{\sqrt{8}} = .005 \text{ cm} \]

c. \( \bar{X} \) is more likely to be within .01 cm of the mean (12 cm) with the second, larger, sample. This is due to the decreased variability of \( \bar{X} \) with a larger sample size.
49. Since \( n = 40 \), the CLT applies to the total time \( T_0 = X_1 + \ldots + X_{40} \). \( T_0 \) has approximately a normal distribution with
\[
\mu_{T_0} = 40(6) = 240 \quad \text{and} \quad \sigma^2_{T_0} = 40(6^2). \quad \text{Thus,} \quad \sigma_{T_0} = 6 \sqrt{40} = 37.95 \quad \text{(all in minutes)}.
\]
   a. Since there are 250 minutes from 6:50 pm to 11:00 pm, we want
   \[
P(T_0 < 250) \approx P(Z < (250 - 240)/37.95) = P(Z < .26) = .6026
   \]
   b. There are 260 minutes until 11:10 pm, so we want
   \[
P(T_0 > 260) \approx P(Z > (260 - 240)/37.95) = P(Z > .53) = 1 - P(Z < .53)
   = 1 - .7019 = .2981.
   \]

*54a \( \mu_x = \mu = 2.65 \)
\[
\sigma_x = \frac{\sigma_x}{\sqrt{n}} = \frac{.85}{5} = .17
\]
\[
P(\bar{X} \leq 3.00) = P\left(Z \leq \frac{3.00 - 2.65}{.17}\right) = P(Z \leq 2.06) = .9803
\]
\[
P(2.65 \leq \bar{X} \leq 3.00) = P(\bar{X} \leq 3.00) - P(\bar{X} \leq 2.65) = .9803 - .5 = .4803
\]

b. If \( P(\bar{X} \leq 3.00) = P\left(Z \leq \frac{3.00 - 2.65}{.85/\sqrt{n}}\right) = .99 \), then \( \frac{.35}{.85/\sqrt{n}} = 2.33 \). Solving this for \( n \) we get \( n = 32.02 \).
   Thus \( n = 33 \) will suffice.

55. \( \mu = np = 20 \quad \sigma = \sqrt{npq} = 3.464 \)
\[
P(25 \leq X) \approx P\left(\frac{24.5 - 20}{3.464} \leq Z\right) = P(1.30 \leq Z) = .0968
\]
\[
P(15 \leq X \leq 25) \approx P\left(\frac{14.5 - 20}{3.464} \leq Z \leq \frac{25.5 - 20}{3.464}\right)
   = P(-1.59 \leq Z \leq 1.59) = .8882
\]

59a. \( E(X_1 + X_2 + X_3) = 180 \)
\( V(X_1 + X_2 + X_3) = 45 \)
\( \sigma_{X_1+X_2+X_3} = 6.708 \)
and \( X_1 + X_2 + X_3 \) is normal.
\[
P(X_1 + X_2 + X_3 \leq 200) = P\left(Z \leq \frac{200 - 180}{6.708}\right) = P(Z \leq 2.98) = .9986
\]
\[ P(150 \leq X_1 + X_2 + X_3 \leq 200) = P(-4.47 \leq Z \leq 2.98) = .9986 - .0000 = .9986 \]

b. \( \mu_X = \mu = 60 \)
\( \sigma_X = \frac{\sigma_X}{\sqrt{n}} = \frac{\sqrt{15}}{\sqrt{3}} = 2.236 \)

and \( \bar{X} \) is normal.
\[ P(\bar{X} \geq 55) = P\left( Z \geq \frac{55 - 60}{2.236} \right) = P(Z \geq -2.236) = .9875 \]
\[ P(58 \leq \bar{X} \leq 62) = P(-.89 \leq Z \leq .89) = .6266 \]

c. Write \( Y = X_1 - .5X_2 - .5X_3 \)
\[ E(Y) = E(X_1 - .5X_2 - .5X_3) = E(X_1) - .5E(X_2) - .5E(X_3) = 0 \]
\[ V(Y) = V(X_1 - .5X_2 - .5X_3) = \sigma_1^2 + .25\sigma_2^2 + .25\sigma_3^2 = 22.5 \quad \text{so} \quad \sigma_Y = 4.7434 \]

And \( Y \) is normal.
\[ P(-10 \leq X_1 - .5X_2 - .5X_3 \leq 5) = P\left( \frac{-10 - 0}{4.7434} \leq Z \leq \frac{5 - 0}{4.7434} \right) \]
\[ = P(-2.11 \leq Z \leq 1.05) = .8531 - .0174 = .8357 \]

d. \( E(X_1 + X_2 + X_3) = 150 \)
\[ V(X_1 + X_2 + X_3) = 36 \]
\[ \sigma_{X_1+X_2+X_3} = 6 \]

and \( X_1 + X_2 + X_3 \) is normal.
\[ P(X_1 + X_2 + X_3 \leq 160) = P\left( Z \leq \frac{160 - 150}{6} \right) = P(Z \leq 1.67) = .9525 \]

We want \( P(X_1 + X_2 \geq 2X_3) = P(X_1 + X_2 - 2X_3 \geq 0) \)
Write \( Y = X_1 + X_2 - 2X_3 \)
\[ E(Y) = E(X_1 + X_2 - 2X_3) = 40 + 50 - 2(60) = -30 \]
\[ V(Y) = V(X_1 + X_2 - 2X_3) = \sigma_1^2 + \sigma_2^2 + 4\sigma_3^2 = 78 \quad \text{so} \quad \sigma_Y = 8.832 \]

And \( Y \) is normal.
\[ P(X_1 + X_2 - 2X_3 \geq 0) = P(Y \geq 0) = P\left( Z \geq \frac{0 - (-30)}{8.832} \right) = P(Z \geq 3.40) = .0003 \]

*64. Let \( X_1, \ldots, X_5 \) denote morning times and \( X_6, \ldots, X_{10} \) denote evening times.
   a. \( E(X_1 + \ldots + X_{10}) = E(X_1) + \ldots + E(X_{10}) = 5E(X_1) + 5E(X_6) \)
   \[ = 5(4) + 5(5) = 45 \]
b. \[ \text{Var}(X_1 + \ldots + X_{10}) = \text{Var}(X_1) + \ldots + \text{Var}(X_{10}) = 5 \text{Var}(X_1) + 5\text{Var}(X_6) \]
\[ = 5 \left[ \frac{64}{12} + \frac{100}{12} \right] = \frac{820}{12} = 68.33 \]

c. \[ \text{E}(X_1 - X_6) = \text{E}(X_1) - \text{E}(X_6) = 4 - 5 = -1 \]
\[ \text{Var}(X_1 - X_6) = \text{Var}(X_1) + \text{Var}(X_6) = \frac{64}{12} + \frac{100}{12} = \frac{164}{12} = 13.67 \]

d. \[ \text{E}[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = 5(4) - 5(5) = -5 \]
\[ \text{Var}[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = \text{Var}(X_1) + \ldots + \text{Var}(X_5) + \text{Var}(X_6) + \ldots + \text{Var}(X_{10}) = 68.33 \]

*72. The total elapsed time between leaving and returning is \( T_o = X_1 + X_2 + X_3 + X_4 \), with
\[ E(T_o) = \sum \mu_i = (15 + 5 + 8 + 12) = 40, \quad \sigma^2_{T_o} = \sum \sigma^2_i = (4^2 + 1^2 + 2^2 + 3^2) = 30, \quad \sigma_{T_o} = 5.477. \]
\( T_o \) is normally distributed, and the desired value \( t \) is the 99\(^{th} \) percentile of the lapsed time distribution added to 10 A.M.: 10:00 + [40+(5.477)(2.33)] = 10:52.76