Problem 1. (15 points) The basketball team is very proud of having recruited Fritz who is 86 inches tall to play center. With Fritz playing, the average height of the starting five is 76.4 inches and the median height is 74 inches. When Fritz is injured, Harry, who is 78 inches tall, has to start at center.

a.) (10 points) When Harry starts, what is the average height of the starting five (in inches)?

\[ 76.4 = \frac{\sum_{i=1}^{5} \text{Height}_i}{5} \]

New Average = \( \frac{(\sum_{i=1}^{5} \text{Height}_i) - \text{Height}_{\text{Fritz}} + \text{Height}_{\text{Harry}}}{5} = \frac{(\sum_{i=1}^{5} \text{Height}_i) - 86 + 78}{5} = \frac{(\sum_{i=1}^{5} \text{Height}_i) - 8}{5} \)

\[ 76.4 - \frac{8}{5} = 76.4 - 1.6 = 74.8 \]

b.) (5 points) When Harry starts, find the median height of the starting five (in inches).

Assuming no ties, there would be 2 heights above 74 and 2 heights below 74 with either Fritz or Harry playing. Thus, the median stays at 74. Ties or not, changing the value of the maximum does not change the median as long as the new value is above the original median.

OVER
Problem 2. (30 points). The random variable $X$ has cdf as follows:

$$
F(x) = \begin{cases} 
0 & \text{if } x < 2 \\
3/10 & \text{if } 2 \leq x < 3 \\
1/2 & \text{if } 3 \leq x < 5 \\
7/10 & \text{if } 5 \leq x < 6 \\
1 & \text{if } 6 \leq x 
\end{cases}
$$

a.) (10 points) Make a table showing the pmf of $X$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>SUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x) = F(x) - F(x^-)$</td>
<td>3/10</td>
<td>2/10</td>
<td>2/10</td>
<td>3/10</td>
<td>1</td>
</tr>
</tbody>
</table>

b.) (8 points) Find $P(2.5 \leq X \leq 5)$.

$$
P(2.5 \leq X \leq 5) = F(5) - F(2.5 - ) = F(5) - F(2) = (7/10) - (3/10) = 0.4 = p(3) + p(5) = 0.2 + 0.2
$$

c.) (12 points) Let $A$ be the event $\{X \leq 5\}$ and $B$ be the event $\{X \geq 2.5\}$.

Are $A$ and $B$ mutually exclusive?

NO $A \cap B = (2.5 \leq X \leq 5)$, which, as we have seen, consists of the points (3.5).

Are $A$ and $B$ independent?

NO $P(A) = F(5) = 0.7$, and $P(B) = 1 - F(2.5 - ) = 1 - F(2) = p(3) + p(5) + p(6) = 0.7$

and $P(A) \times P(B) = 0.49 \neq P(A \cap B) = 0.4$
Problem 3. (30 points). We have two urns. In urn 1, there are 7 red balls and 5 green balls. In urn 2, there are 8 red balls and 4 green balls. Consider a two step experiment. In step 1, we select one of the urns. Let \( A \) be the event that we select urn 1. We are given that \( P(A) = 0.6 \).

In step 2, we choose 3 balls at random from the urn selected in step 1.

Let \( B \) be the event that we choose 1 red ball and 2 green balls at step 2.

a.) (18 points) Find \( P(A \cap B) \), \( P(A' \cap B) \), and \( P(B) \).

\[
P(B | A) = \frac{\binom{7}{1} \binom{5}{2}}{\binom{12}{3}} = \frac{7 \times \frac{5 \times 4}{2}}{12 \times 11 \times 10} \cdot \frac{3 \times 2}{220} = \frac{70}{220} = \frac{7}{22}
\]

\[
P(A \cap B) = P(A) \times P(B | A) = (0.6)(0.318) = 0.191
\]

\[
P(B | A') = \frac{\binom{8}{1} \binom{4}{2}}{\binom{12}{3}} = \frac{8 \times \frac{4 \times 3}{2}}{12 \times 11 \times 10} \cdot \frac{3 \times 2}{220} = \frac{48}{220} = \frac{6}{22}
\]

\[
P(A' \cap B) = P(A') \times P(B | A') = (0.4)(0.218) = 0.087
\]

\[
P(B) = P(A \cap B) + P(A' \cap B) = 0.278
\]

b.) (12 points) Find \( P(C | A) \).

\[
P(C | A) = P(3 \text{ Red} | A) + P(3 \text{ Green} | A)
\]

\[
= \binom{7}{3} \binom{5}{0} \frac{7 \times 6 \times 5}{220} + \binom{7}{3} \binom{0}{2} \frac{5 \times 4}{220} = \frac{35}{220} + \frac{45}{220} = \frac{5}{44} = 0.205
\]

Let \( C \) be the event that all three balls chosen at step 2 are of the same color.

OVER
Problem 4 (25 points). Let \( A \) be the event that a book on statistics by Mode is presently checked out of the engineering library, and let \( B \) be the event that a book on statistics by Mean is presently checked out.

a.) (9 points). If \( P(A) = 0.5 \), \( P(B) = 0.3 \), and \( P(A \cup B) = 0.6 \), find \( P(A \cap B) \).

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow 0.6 = 0.5 + 0.3 - P(A \cap B) \Rightarrow P(A \cap B) = 0.8 - 0.6 = 0.2
\]

b.) (9 points). Using the probabilities in part (a), find \( P(\text{exactly one of the two books is checked out}) \).

\[
P(\text{exactly one of the two books is checked out}) = P(A \cup B) - P(A \cap B) = 0.6 - 0.2 = 0.4
\]

Alternatively: \( \text{(exactly one of the two books is checked out)} = (A' \cap B) \cup (A \cap B') \), so that

\[
P(\text{exactly one of the two books is checked out}) = P(A' \cap B) + P(A \cap B'), \text{ and}
\]

\[
P(A' \cap B) = P(B) - P(A \cap B) = 0.3 - 0.2 = 0.1
\]

\[
P(A \cap B') = P(A) - P(A \cap B) = 0.5 - 0.2 = 0.3
\]

c.) (7 points). If \( P(A \cup B) = 0.9 \), \( P(A \cap B) = 0.3 \), and \( P(\text{only the book by Mode is checked out}) = 0.4 \), find \( P(\text{only the book by Mean is checked out}) \).

Note that \( \text{(only the book by Mode is checked out)} = (A \cap B') \), and that

\[
\text{(only the book by Mean is checked out)} = (A' \cap B)
\]

\[
P(A \cup B) = P(A \cap B') + P(A' \cap B) + P(A \cap B)
\]

\[
0.9 = 0.4 + P(A' \cap B) + 0.3 \Rightarrow P(A' \cap B) = 0.2
\]

ALTERNATIVE 1:

\[
P(\text{exactly one of the two books is checked out}) = P(A \cup B) - P(A \cap B) = 0.9 - 0.3 = 0.6 = P(A' \cap B) + P(A \cap B') = P(A' \cap B) + 0.4 \Rightarrow P(A' \cap B) = 0.6 - 0.4 = 0.2
\]

ALTERNATIVE 2 (The long way):

\[
P(A) = P(A \cap B') + P(A \cap B) = 0.4 + 0.3 = 0.7
\]

\[
P(B) = P(A \cup B) - P(A) + P(A \cap B) = 0.9 - 0.7 + 0.3 = 0.5
\]

\[
P(A' \cap B) = P(B) - P(A \cap B) = 0.5 - 0.3 = 0.2
\]