ELASTIC SHAPE ANALYSIS OF SURFACES

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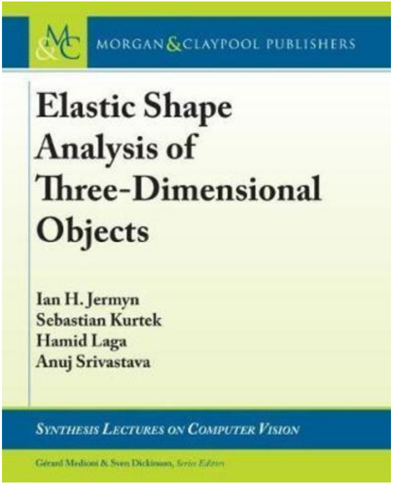
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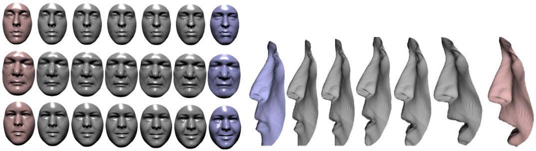
Outline

Outline

Motivation Examples - Human Body

- Human biometrics is a fascinating problem area.
- Facial Surfaces: 3D face recognition for biometrics

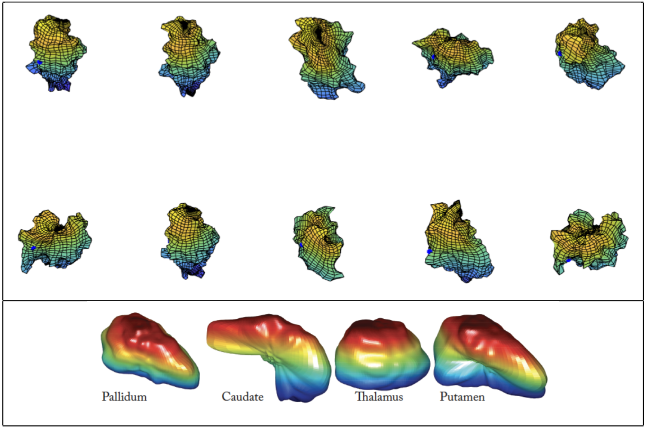


- Human bodies: applications – medical (replace BMI), textile design.

- Shapes are represented by surfaces in $\mathbb{R}^3$

Motivation Examples - Biomolecules

- Protein surfaces, anatomical surfaces



- Shapes are represented by surfaces in $\mathbb{R}^3$
- Goal is to cluster and classify proteins, and study structure versus functionality.

Broad Goals

- Assume all the objects have the same topology, as described below.
- They are all maps of the type: $f : D \to \mathbb{R}^3$, where D is a two-dimensional compact space. Examples:
  - $D = [0,1]^2$: f is called a quadrilateral surface.
  - $D = \mathbb{S}^2$: f is called a spherical surface.
  - $D = \mathbb{S}^1 \times [0,1]$: f is called a disc surface.
- We say that f is an immersion if the differential $df_s : \mathbb{R}^2 \to \mathbb{R}^3$ is full ranked. We will assume that all our surfaces are immersions.
- We will denote $s = (u, v)$ to be the coordinates in D. Being an immersion implies that $\frac{\partial f}{\partial u}$ is not a multiple of $\frac{\partial f}{\partial v}$. In other words, the cross product $\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$ is not zero.

Broad Goals

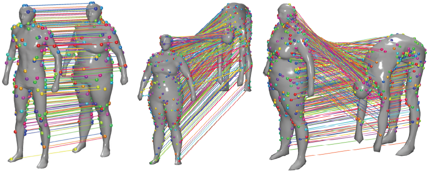
- Metric: A quantification of differences between shapes of given 3D objects. The metric should be invariant to all the desired shape preserving transformations.

How different are these shapes?
$\Longleftrightarrow$

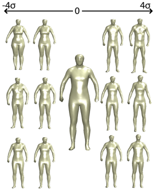
- Geodesic: Optimal deformation of one shape into the other.

Broad Goals

- Registration: Given any two objects find a mapping that assigns each point on an object to a unique point on another object.



- Summary: Ability to compute representative shapes (mean, median), and to study the dominants modes of variability (covariance, PCA) in a given set of shapes.



Broad Goals

- Clustering and Classification: Unsupervised and supervised classification of shapes.

- Shape Regression: Use shape as a predictor for predicting scalar or vector response variables.

Outline

Desired Invariances

Focus on the maps: $f : \mathbb{S}^2 \to \mathbb{R}^3$ that are immersions.

- Translation group action: $f(s) \mapsto f(s) + x, \quad x \in \mathbb{R}^3$
- Rotation group action: $f(s) \mapsto Of(s), \quad O \in SO(3)$
- Scaling group action: $f(s) \mapsto af(s), \quad a \in \mathbb{R}_+$
- Re-parameterization group actions: $f(s) \mapsto f(\gamma(s)), \ \gamma \in \Gamma$. Here Γ is the group of orientation-preserving diffeomorphisms of $\mathbb{S}^2$. Then, for any $\gamma \in \Gamma$, and a surface f, the composition $f \circ \gamma$ is simply a re-parameterization of f. It does not change the shape of f.

Want a metric such that

$$\boxed{d_s(f_1, f_2) = d_s(a_1 O_1 f_1(\gamma_1(s)) + x_1, a_2 O_2 f_2(\gamma_2(s)) + x_2)} \ .$$

Looking for representing f for the purpose of shape analysis.

- f is not invariant to any of the desired transformations.
- Gradient Field: The gradient $\nabla f : \mathbb{S}^2 \to \mathbb{R}^{3\times 2}$, is $\nabla f(s) = [f_u, f_v]$. It is a 3×2 matrix at each point $s \in \mathbb{S}^2$. Here $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$.

  ∇f is invariant to translation of f but not to rotation, scale, and re-parametrization. One can reconstruct f from ∇f as it has all the information about f.
- Normal Vector Fields: For $s = (u, v) \in \mathbb{S}^2$, the normal vector field is $\tilde{n}(s) = \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$. The unit normal vector field is $n(s) = \frac{\tilde{n}(s)}{|\tilde{n}(s)|}$.

  $n(s)$ is invariant to translation of f but not rotation and re-parametrization. One cannot reconstruct f from n as it does not have all the information about f.

Mathematical Representations of Spherical Surfaces

- Tensor Field: Form a tensor field (induced metric) on $\mathbb{S}^2$ is:

$$g(s) = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \nabla f(s)^T \nabla f(s) = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

  It is called the First Fundamental Form of f. g is invariant to translation and rotation of f but not re-parametrization.

  One cannot reconstruct f from g only as it has partial information about f.

- Area Element: The area element $r(s) = |\tilde{n}(s)| = \sqrt{\det(g(s))}$. It contains partial information about f.

Closer Look

- Metric Tensor Field on $\mathbb{S}^2$:
  - Take a closer look at $g(s)$, $s \in \mathbb{S}^2$. g is a tensor field on $\mathbb{S}^2$. It specifies a Riemannian metric on $\mathbb{S}^2$ – called a *pullback metric*.
  - For any spherical curve $\beta : [0, 1] \to \mathbb{S}^2$, we can compute its length:

$$L[\beta] = \int_0^1 \left\langle\langle \dot{\beta}(s), \dot{\beta}(s) \right\rangle\rangle_{\beta(s)} ds = \int_0^1 (\dot{\beta}(s)^T g(s) \dot{\beta}(s)) ds$$

    It is the Euclidean length of the corresponding curve $f(\beta(s))$, the corresponding curve on f.
  - The space of all g's is the set of all Riemannian metrics on $\mathbb{S}^2$, also denoted by $\mathrm{Met}(\mathbb{S}^2)$.
- Gauss Map: The mapping $s \in \mathbb{S}^2 \mapsto n(s) \in \mathbb{S}^2$ is called the Gauss map of f. The set of all Gauss maps is $\mathcal{C}^\infty(\mathbb{S}^2, \mathbb{S}^2)$.

Higher-Order Representations

- Representations involving second derivatives. They are invariant to rotation and translation, but not to re-parameterizations. It is difficult to reconstruct a surface given these representations.
- The second fundamental form is given by

$$\Pi = L\, du^2 + 2M\, du\, dv + N dv^2 \;,$$

where

$$L = f_{uu} \cdot n, \quad M = f_{u,v} \cdot n, \quad N = f_{v,v} \cdot n \;.$$

We can also write:

$$\Pi = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \;.$$

- Shape Operator: At each point $s \in \mathbb{S}^2$, the matrix $S = g^{-1}\Pi \in \mathbb{R}^{2\times 2}$ is called the shape operator of f at $f(s)$.
- Let λ_1, λ_2 be the eigenvalues of S. Then,
  - $\lambda_1\lambda_2 = \det(S)$ is the Gaussian curvature of the surface f at $f(s)$.
  - $\lambda_1 + \lambda_2 = \mathrm{trace}(S)$ is the mean curvature of the surface f at $f(s)$.

Chosen Mathematical Representation

- We will use the pair (g, n) to represent a surface f for the purpose of shape analysis.
  Let $\Phi : f \mapsto \Phi(f) = (g, n) \in \mathrm{Met}(\mathbb{S}^2) \times \mathcal{C}^\infty(\mathbb{S}^2, \mathbb{S}^2)$.
- Effect of group actions:
  - Translation: $f \mapsto f + x$, $(g, n) \mapsto (g, n)$.
  - Rotations: $f \mapsto Of$, $(g, n) \mapsto (g, On)$.
  - Scaling: $f \mapsto af$, $(g, n) \mapsto (a^2 g, n)$.
  - Re-parametrization: $f \mapsto f \circ \gamma$, $(g, n) \mapsto (J_\gamma^T (g \circ \gamma) J_\gamma, n \circ \gamma)$.
- If the surface f is perturbed to $f + \delta f$, then the corresponding g and n are changed by the quantities: [Tumpach et al. TPAMI 2017]:

$$
\begin{aligned}
\delta g &= \mathrm{Jac}(f)^T \,\mathrm{Jac}(\delta f) + (\mathrm{Jac}(\delta f))^T \,\mathrm{Jac}(f) \\
&= \begin{pmatrix} 2 f_u \cdot \delta f_u & f_u \cdot \delta f_v + f_v \cdot \delta f_u \\ f_u \cdot \delta f_v + f_v \cdot \delta f_u & 2 f_v \cdot \delta f_v \end{pmatrix}, \\
\delta n &= -\frac{1}{2} \mathrm{Tr}(g^{-1} \delta g)\, n + \frac{1}{|g|^{\frac{1}{2}}} \left(\delta f_u \times f_v + f_u \times \delta f_v \right).
\end{aligned}
$$

This is the differential of Φ.

Elastic Riemannian Metric

- Let $(\delta g_1, \delta n_1)$ and $(\delta g_2, \delta n_2)$ represent two tangent vectors of the representation space at the point (g, n).
- Elastic Riemannian Metric: Define the Riemannian metric

$$\langle\langle(\delta g_1, \delta n_1), (\delta g_2, \delta n_2)\rangle\rangle_{(g,n)} =$$

$$a \int_{\mathbb{S}^2} Tr(g^{-1}\delta g_1 g^{-1}\delta g_2) r(s) ds \quad \text{— Term 1}$$

$$+b \int_{\mathbb{S}^2} Tr(g^{-1}\delta g_1) Tr(g^{-1}\delta g_2) r(s) ds \quad \text{— Term 2}$$

$$+c \int_{\mathbb{S}^2} \langle \delta n_1, \delta n_2 \rangle \, r(s) ds \quad \text{— Term 1}$$

where a, b, and c are positive weights.

Elastic Riemannian Metric

- Elastic Riemannian Metric: One can rearrange the terms to reach:

$$
\begin{aligned}
&\langle\langle(\delta g_1, \delta n_1), (\delta g_2, \delta n_2)\rangle\rangle_{(g,n)} = \\
&a \int_{\mathbb{S}^2} Tr((g^{-1}\delta g_1)_0 (g^{-1}\delta g_2)_0) r(s) ds \quad \text{— Term 1} \\
&+\tilde{b} \int_{\mathbb{S}^2} Tr(g^{-1}\delta g_1) Tr(g^{-1}\delta g_2) r(s) ds \quad \text{— Term 2} \\
&+c \int_{\mathbb{S}^2} \langle \delta n_1, \delta n_2 \rangle \, r(s) ds \quad \text{— Term 1}
\end{aligned}
$$

where $\tilde{b} = a/2 + b$.
Here $A_0 = A - \text{trace}(A) I_{2\times 2}$ denotes the traceless part of a matrix A.

Interpretations of the terms

- Stretching Term:
  - Let $A \mapsto \det(A)$ be the determinant function, then $\mathrm{trace}(A^{-1}H)$ is the directional derivative of $\det(\cdot)$ in the direction of H.
  - Similarly, $\mathrm{trace}(g^{-1}\delta g_i)$ is the derivative of $\det(\cdot)$ at g in the direction of δg_i.
  - Since $\det(g)$ is a measure of the area of a small patch at $f(s)$, this term measures the change in area of this patch.
- Bending Term: δn_1, δn_2 are the changes in the direction normal to the patch. Thus, the third term measures the change in the rotation of that patch.
- Shape Term: In the first term, we can removed translation, rotation, and scale. The only information that is left is about the shape of the patch. Thus, the first term measures a change in the shape of that patch.

Invariance Properties

- Translation Group: (g, n) are invariant to the translation of f.
- Scaling Group: The surface area of f equals $A = \int_{\mathbb{S}^2} r(s)ds$. We can rescale the surfaces by $f \mapsto f/\sqrt{A}$. Now, the surfaces is rescaled to unit area and is invariant to scale variability.
- Re-Parameterization Group:
  This metric is preserved under the action of re-parameterization group. That is:

$$
\begin{aligned}
& \langle\langle(\delta g_1, \delta n_1), (\delta g_2, \delta n_2)\rangle\rangle_{(g,n)} \\
= \; & \langle\langle(\delta(g_1 \circ \gamma), \delta(n_1 \circ \gamma)), (\delta(g_2 \circ \gamma), \delta(n_2 \circ \gamma))\rangle\rangle_{((g\circ\gamma),(n\circ\gamma))}
\end{aligned}
$$

- Rotation Group: If $\Phi(f) = (g, n)$, then $\Phi(Of) = (g, On)$ for any $O \in SO(3)$.

$$
\langle\langle(\delta g_1, \delta n_1), (\delta g_2, \delta n_2)\rangle\rangle_{(g,n)} = \langle\langle(\delta g_1, O\delta n_1), (\delta g_2, O\delta n_2)\rangle\rangle_{(g,On)}
$$

  This metric is preserved under the action of the rotation group.
- Challenge: Metric is too complex to work with directly.

Square-Root Normal Field

- The SRNF of $f \circ \gamma$ is given by $(q, \gamma) \equiv (q \circ \gamma) J_\gamma$, where J_γ is the determinant of the Jacobian of γ.
- This representation satisfies the isometry condition:

$$\|q_1 - q_2\| = \|(q_1, \gamma) - (q_2, \gamma)\|, \;\; \forall q_1, q_2 \in \mathbb{L}^2, \; \gamma \in \Gamma \; .$$

- The shape metric is given by:

$$d_s([q_1], [q_2]) = \inf_{(O,\gamma) \in SO(n) \times \Gamma} \|q_1 - O(q_2, \gamma)\| \; .$$

This include rotational alignment and non-rigid registration of surfaces.

Potential Representations of Surfaces

	Represent. 1	Represent. 2	Represent. 3	Represent. 4
	Surface	Gradient	Tensor + Normal	SRNF
Symbol	f	∇f	$(g = \nabla f^T \nabla f, n = \nabla f^\perp)$	$q = \sqrt{r} n$
Invariance	None	Translation	Translation (Rotation)	Translation
Elastic Metric	Complicated	Complicated	Relat. Simple	Simple
Geodesic	Complicated	Complicated	Relat. Simple	Simple
Registration	Complicated	Complicated	Relat. Simple	Simple
Reconstruction	-	Trivial	Relat. Difficult	Difficult

Potential Representations of Surfaces

The elastic metric is defined in the Representation 3 space (g, n).

- Representation 1 – f: One can use a pullback metric in the original surface space $\mathcal{F}$ to define and compute geodesics.
  - Pros: No need to reconstruct.
  - Cons: Operations (geodesic, registration, etc) are computationally expensive.
- Representation 2 – ∇f: One can use a pullback metric in the space of vector fields to define and compute geodesics.
  - Pros: Reconstruction is trivial.
  - Cons: Operations (geodesic, registration, etc) are somewhat expensive. Not all vector fields are gradient vector fields.
- Representation 3 – (g, n): Metric is defined on this space directly.
  - Pros: Geodesic are available analytically.
  - Cons: Registration is still a challenge. Reconstruction is difficult now. Not all tensor fields are forward maps of surfaces.
- Representation 4 – q: Partial elastic metric is $\mathbb{L}^2$
  - Pros: Geodesics are trivial. Registration under the $\mathbb{L}^2$ norm is the simplest.
  - Cons: Reconstruction is most difficult now. Not all normal fields are forward maps of surfaces.

Outline

SRNF Inversion: A Numerical Approach

- Similar to curves, all the analysis takes place in the SRNF space. Final solutions need to be inverted back to the original space.
- Don't know enough about the properties of the map $Q : f \to q$. Q is not injective, and not surjective. (Perhaps locally injective up to certain transformations.)
- We take a numerical approach to solve the inverse problem.

$$\hat{f} = \underset{f \in \mathcal{F}}{\text{arginf}} \, \|Q(f) - q_0\|^2$$

  Even if the inverse does not exist, we find a surface whose SRNF is as close as possible to the given q_0.

  - If there exists an f such that $Q(f) = q_0$, then the procedure will find it.
  - If there exist several fs such that $Q(f) = q_0$ for each of these surfaces, then the procedure will find one such f.
  - If there does not exist any f such that $Q(f) = q_0$, then the procedure will find an f that minimizes this cost function. In other words, a surface whose SRNF comes as close to the given q_0 as possible.

SRNF Inversion: A Numerical Approach

- Modify the cost function to:

$$\hat{f} = \operatorname*{arginf}_{f \in \mathcal{F}} E[w; q_0, f_0], \quad E[w; q_0, f_0] = \|Q(f_0 + w) - q_0\|^2$$

  where f_0 is an initial guess and w is the deformation.

- Let $\mathcal{B} = \{b_1, b_2, \dots\}$ represents an orthogonal basis for representing surfaces w, f. Then, the directional derivative of E in the direction of a basis element b is given by:

$$\begin{aligned} \nabla_{b_j} E(w; f_0, q_0) &= \frac{d}{d\epsilon} \|Q(f_0 + w + \epsilon b_j) - q\|^2 \\ &= 2 \left\langle Q(f_0 + w) - q, dQ_{f_0+w}(b_j) \right\rangle \end{aligned}$$

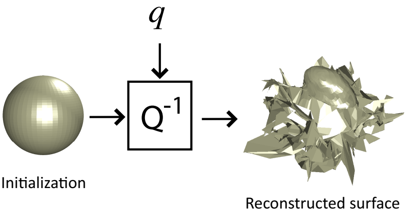
  Here $dQ_{f_0+w}(b_j) = \frac{1}{\sqrt{|n|}} \left(\tilde{n}_b - \frac{n \cdot \tilde{n}_b}{2} n \right)$, $\tilde{n}_b = (f_u \times b_{j,v}) + (b_{j,u} \times f_v)$.

- The full gradient for optimization over w is given by:

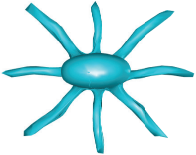
$$\nabla E = \sum_{j=1}^{J} (\nabla_{b_j} E(w; f_0, q_0)) b_j \ .$$

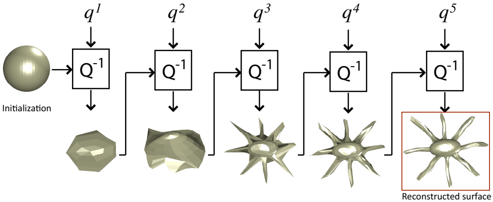
SRNF Inversion: A Numerical Approach

- Solution at the highest resolution:



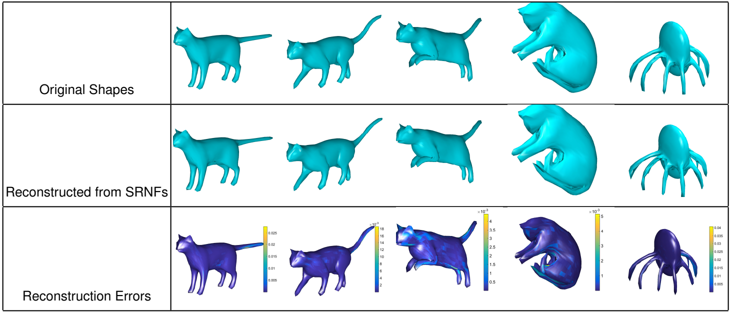
- Developed a multi-resolution approach to solve this optimization problem.
- Deform a unit sphere (f_0) using smaller and then higher resolution.



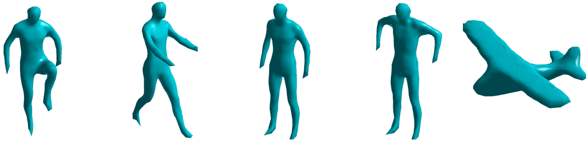


SRNF Inversion

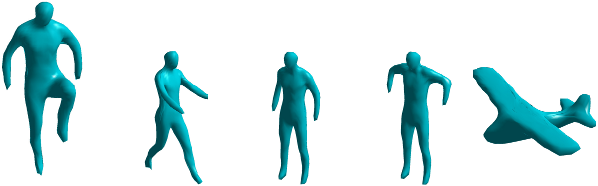
- Some examples:



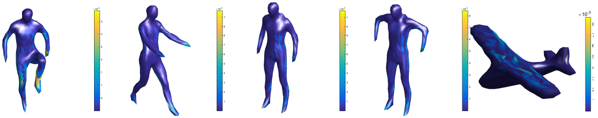
SRNF Inversion



(a) The target surfaces f_o.



(b) The reconstructed surfaces f^*.



(c) Reconstruction errors $|f^*(s) - f_o(s)|$.

In these examples we used 3642 spherical harmonic basis elements.

Shape Registration

- Registration optimization:

$$\inf_{(O,\gamma)\in SO(n)\times\Gamma} \|q_1 - O(q_2,\gamma)\|$$

- Optimization using: Procrustes rotation (for O) and Lie algebra of Γ (for γ).
- Γ is an infinite-dimensional, nonlinear manifold. To optimize a cost function over it is complicated.

Shape Registration: Rotation

Procrustes rotation assuming a given registration (γ). Let $\tilde{f}_2 = f_2 \circ \gamma$.

1. Compute the SRNFs $q_1 = Q(f_1)$ and $\tilde{q}_2 = Q(\tilde{f}_2)$.
2. Compute the 3×3 matrix $A = \int_{\mathbb{S}^2} q_1(s)\tilde{q}_2(s)^T \, ds$.
3. Compute the singular value decomposition $A = U\Sigma V^T$.
4. Compute the optimal rotation as $O^* = UV^T$. (If the determinant of A is negative, the last column of V changes sign.)
5. Compute the optimally rotated surface $\tilde{f}_2^* = O^* \tilde{f}_2$.

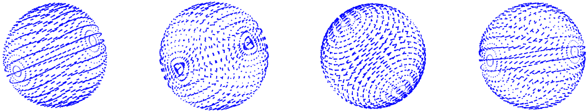
Shape Registration: Re-parameterization

- Divide the search for optimal γ into a number of smaller γs.
- Define the cost function at the k^{th} iteration to be:

$$H(\gamma) = \|q_1 - ((q_2, \gamma_k), \gamma)\|^2 \ ,$$

where γ_k is the total diffeomorphism at that stage for registration.

- We know that: $T_{\gamma_{id}}(\Gamma) = \{w \ \text{ is a smooth vector field on } \ \mathbb{S}^2\}$. We can compute an orthonormal basis of this set; Call it $\mathcal{B} = \{b_1, b_2, \dots\}$. γ is assumed to be a small diffeo (close to identity).



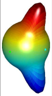
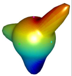
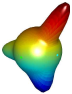
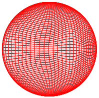
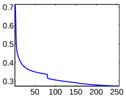
- Then, for any functional $H : \Gamma \to \mathbb{R}$, we can express: $\nabla_{\gamma_{id}} H = \sum_j c_j b_j$, where c_j is the directional derivative of H in the direction of b_j.

Shape Registration: Re-parameterization

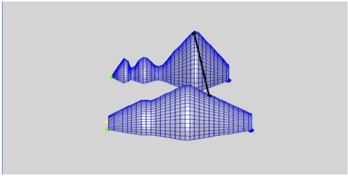
- We compute the directional derivative using:

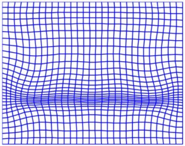
$$c_j = \frac{1}{\epsilon}\left(H(\gamma_{id} + \epsilon b_j) - H(\gamma_{id})\right) .$$

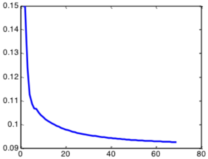
- Compute the incremental diffeomorphism: $\gamma^{k+1} = \exp_s(w(s))$, $w(s) = \sum_j c_j b_j(s)$ and the exponenital map is on $\mathbb{S}^2$.
- Compute the cumulative diffeomorphism: $\gamma_{k+1} = \gamma_k \circ \gamma^{k+1}$.
- Example:

				
f_1	f_2	$O^*(f_2 \circ \gamma^*)$	γ^*	E

Registration Example

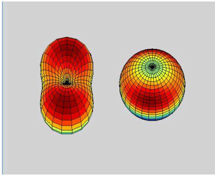


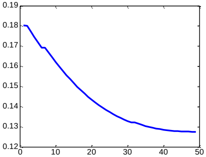




Registration movie:

Registration Example



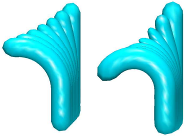


Registration movie:

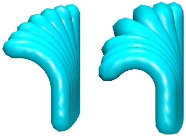
Outline

Shape Geodesics

- Given two surfaces f_1 and f_2.
- Compute their SRNFs q_1 and 2_2.
- Solve for the optimal registration $(O^*, \gamma^*) = \inf_{(O,\gamma)} \|q_1 - O(q_1.\gamma)\|$.
- Draw a straight line between q_1 and $q_2^* = O^*(q_2, \gamma^*)$. Take each point along this path and use SRNF inversion to find the corresponding surface.



(a) Linear path, $(1-t)f_1 + tf_2^*$.



(b) Geodesic path by SRNF inversion.

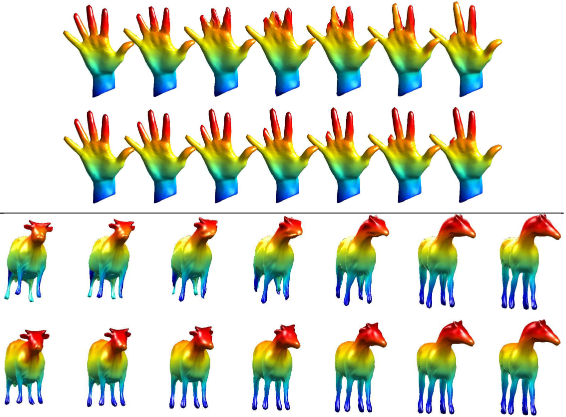
Shape Registration and Geodesics

Without Reg.

With Reg.

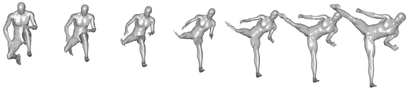
Without Reg.

With Reg.

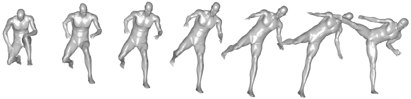


Geodesics are computed in the SRNF space and then each point along the path is inverted back numerically.

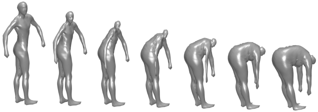
Elastic Geodesic Examples



(a) Linear path, $(1 - t)f_1 + tf_2^*$. The registration has been computed using SRNFs.



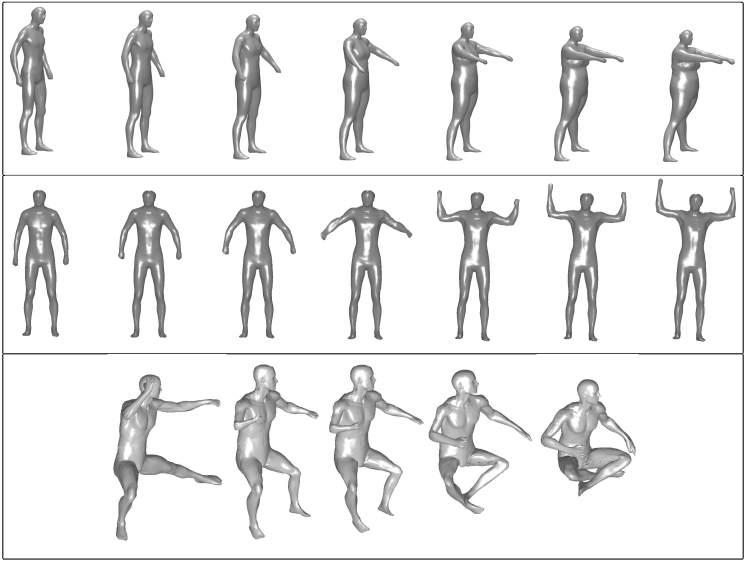
(b) Geodesic path after registration using SRNF inversion.



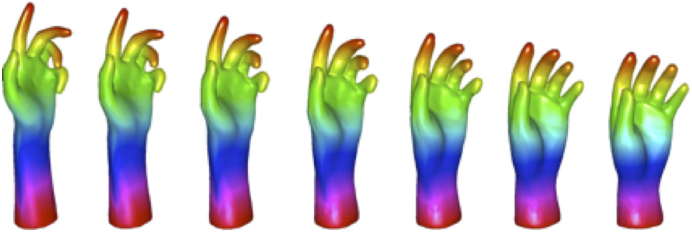
(a) The linear path, $(1 - t)f_1 + tf_2^*$. The registration has been computed using SRNFs.

(b) Geodesic path after registration using SRNF inversion.

Elastic Geodesic Examples

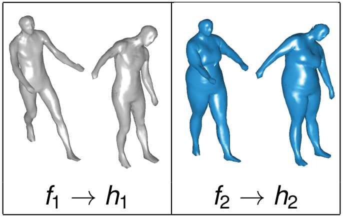


Elastic Geodesic Examples

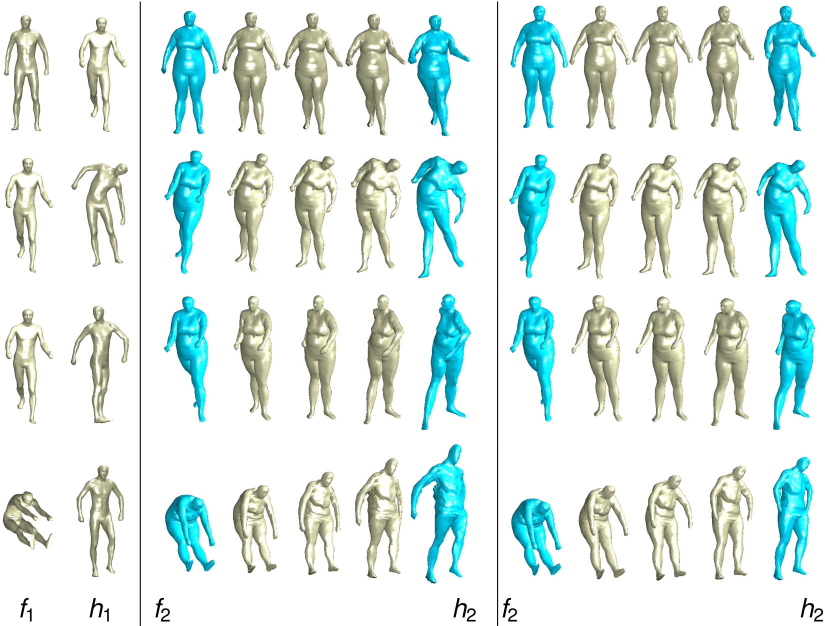


Deformation Transfer Using Parallel Transport

- Goal: Compute the deformation between two given surfaces, and apply them to a third surface.
- Approach:
  - Compute the deformation vector field v from f_1 to h_1 by computing a geodesic path between their shapes.
  - Transfer v at f_1 to f_2 using parallel transport resulting in $v^{||}$.
  - Deform f_2 into h_2 using the exponential map of $v^{||}$.



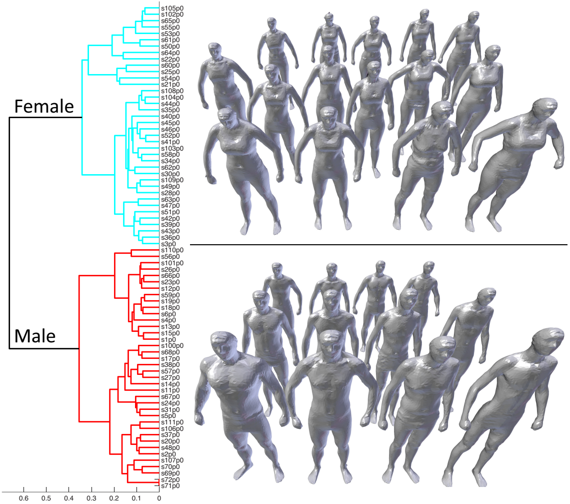
Deformation Transfer



(a) Source (b) Deformation transfer by linear (c) Deformation transfer by

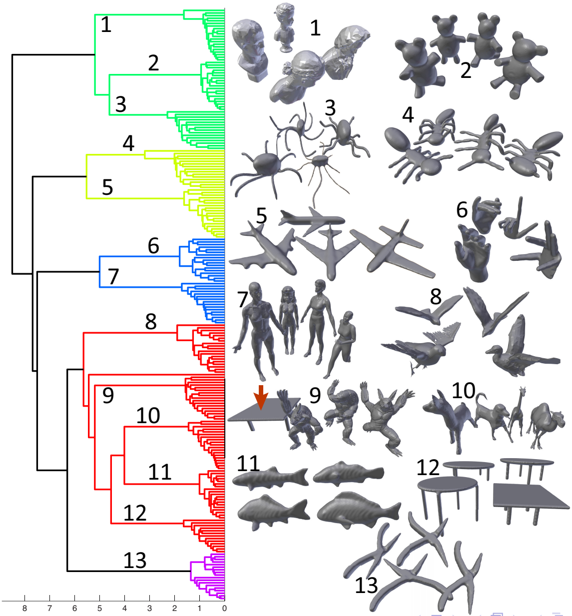
Shape Clustering

Use elastic shape metric for clustering.



Shape Clustering

Use elastic shape metric for clustering.



Shape-Based Classification

- Use MRI scans of human brains to extract certain subcortical structures of interest.
- Scan data is available for subjects that have ADHD and that are normal subjects.
- Want to use the shapes of these structures to classify subjects in ADHD and normal class.

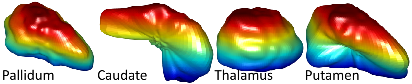


Figure: Left subcortical structures in the brain.

Table: Classification performance, in %, for six different techniques, namely the Gaussian classifier on the (1) SRNF and (2) SRM spaces, and the Nearest Neighbor (NN) classifier using the (3) SRM space, (4) Harmonic method, (5) ICP algorithm, and (6) SPHARM-PDM.

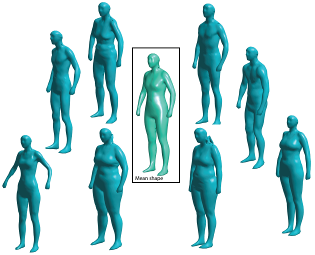
	SRNF Gauss	SRM Gauss	SRM NN	SPHARM	ICP	SPHARM-PDM
L. Caudate	**67.7**	-	41.2	64.7	32.4	61.8
L. Pallidus	85.3	**88.2**	76.5	79.4	67.7	44.1
L. Putamen	**94.1**	82.4	82.4	70.6	61.8	50.0
L. Thalamus	**67.7**	-	58.8	**67.7**	35.5	52.9
R. Caudate	55.9	-	50.0	44.1	50.0	**70.6**
R. Pallidus	**76.5**	67.6	61.8	67.7	55.9	52.9
R. Putamen	67.7	**82.4**	67.7	55.9	47.2	55.9
R. Thalamus	**67.7**	-	58.8	52.9	64.7	64.7

Shape Summaries

Sample mean:

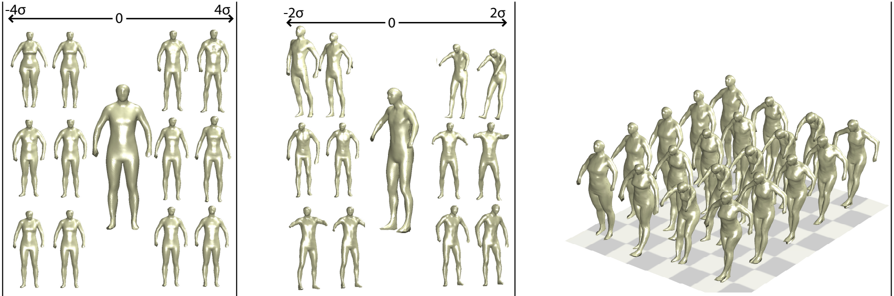
$$\mu_q = \underset{[q]\in\mathcal{S}}{\operatorname{argmin}} \sum_{i=1}^{n} d_s([q],[q_i])^2$$

Then, $\mu_q \mapsto \mu_f$ (SRNF Inversion).



Shape PCA and Modeling

Use the tangent bundle of shape spaces to perform PCA and wrap it back on the shape space to study principal directions.



(a) Mean shape and its first three modes of variation.

(b) Mean pose and its first three modes of variation.

(c) Random samples from the PCA model on $\mathcal{S}$.

Random Shape Models

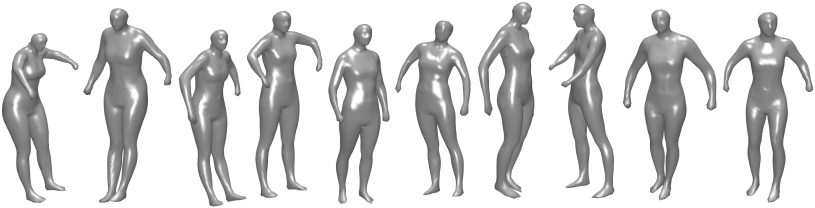


Figure: Ten arbitrary 3D human body shapes automatically synthesized by sampling from a Gaussian distribution fitted, in the SRNF shape space $\mathcal{S}$, to a collection of human body shapes belonging to different subjects in different poses.

Outline

Quantify Reflection Symmetry

Let $v \in \mathbb{R}^3$ be a vector orthogonal to a plane. Then:

$$\tilde{f} = H(v)f, \text{ where } H(v) = \left(I - 2\frac{vv^T}{v^T v} \right). \tag{1}$$

$H(v)$ is the reflection matrix for the plane perpendicular to the vector v. f is the original surface and $\tilde{f}$ is its reflection under the chosen plane. Let α be a geodesic in between f and $\tilde{f}$ in the shape space.

- First, its length gives a formal measure of asymmetry of f.
- Second, the halfway point along this geodesic, *i.e.* $\alpha(0.5)$, is symmetric.
- Lastly, if this geodesic path is unique, then amongst all symmetric shapes, $\alpha(0.5)$ is the nearest to f in $\mathcal{F}$ under the shape metric. The path from $\alpha(0)$ to $\alpha(0.5)$ is precisely the smallest deformation needed to symmetrize f. Thus, as already stated, half of the length of this path is also a measure of asymmetry of the shape.

Quantify Reflection Symmetry

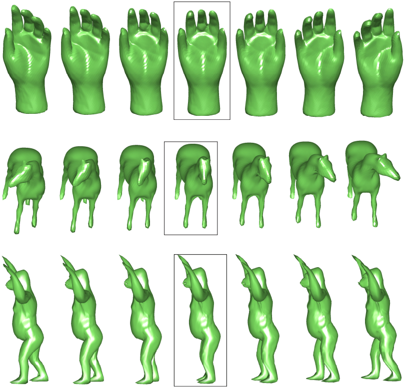


Figure: Symmetrizing complex surfaces. Each example shows the geodesic between a surface and its reflection. The highlighted midpoint of the geodesic is the nearest symmetric shape.

Summary: Shape Analysis of Surfaces

- For registration of points across surfaces one needs an invariant Riemannian metric, leading to an invariant distance.
- This metric is too complex to be useful in practical situations. A square-root transformation, SRNF, converts this metric into a simpler $\mathbb{L}^2$ metric.
- We define quotient spaces of $\mathbb{L}^2$ under shape-preserving transformations, such as the rotation and re-parameterizations.
- All the operations – registration, geodesics, statistical analysis, etc. – take place in the SRNF space. Final solutions are converted back to surface space by inverting SRNFs.